



MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG



COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Automatic Generation of Minimal and Reduced Models for Structured Parametric Dynamical Systems

Igor Pontes Duff

Joint work with

Peter Benner (MPI, Magdeburg)

Pawan Goyal (MPI, Magdeburg)

ICERM Workshop - Mathematics of Reduced Order Models

Providence, RI, USA

February 20, 2020



Table of Contents

1. Introduction
2. First-order systems
3. Structured Transfer Function
4. Parametric extension
5. Numerical Examples
6. Outlook

Continuous mechanics

$$\vec{\nabla} \cdot \sigma = \rho \frac{\partial^2 s}{\partial t^2}$$

$$\varepsilon = \frac{1}{2} (\nabla s + \nabla^T s), \quad \sigma = \lambda \operatorname{tr}(\varepsilon) \mathbf{I} + 2\mu \varepsilon$$



CSC

Introduction

–Large-Scale Dynamical Systems–

Continuous mechanics

$$\vec{\nabla} \cdot \sigma = \rho \frac{\partial^2 s}{\partial t^2}$$

$$\varepsilon = \frac{1}{2} (\nabla s + \nabla^T s), \quad \sigma = \lambda \operatorname{tr}(\varepsilon) \mathbf{I} + 2\mu\varepsilon$$

semi-discretization in space

High order ODE

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) &= \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned}$$



Introduction

–Large-Scale Dynamical Systems–

Continuous mechanics

$$\vec{\nabla} \cdot \sigma = \rho \frac{\partial^2 s}{\partial t^2}$$

$$\varepsilon = \frac{1}{2} (\nabla s + \nabla^T s), \quad \sigma = \lambda \operatorname{tr}(\varepsilon) \mathbf{I} + 2\mu\varepsilon$$

semi-discretization in space

High order ODE

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Heated rod with delay

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}(x, t) + a_0(x)v(x, t) + a_1(x)v(x, t - 1)$$

$$v(0, t) = u(t) \quad t \geq 0$$



Introduction

–Large-Scale Dynamical Systems–

Continuous mechanics

$$\vec{\nabla} \cdot \sigma = \rho \frac{\partial^2 s}{\partial t^2}$$

$$\varepsilon = \frac{1}{2} (\nabla s + \nabla^T s), \quad \sigma = \lambda \text{tr}(\varepsilon) \mathbf{I} + 2\mu\varepsilon$$

semi-discretization in space

High order ODE

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Heated rod with delay

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}(x, t) + a_0(x)v(x, t) + a_1(x)v(x, t - 1)$$
$$v(0, t) = u(t) \quad t \geq 0$$

semi-discretization in space

Functional differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_\tau\mathbf{x}(t - \tau) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$



Continuous mechanics

$$\vec{\nabla} \cdot \sigma = \rho \frac{\partial^2 s}{\partial t^2}$$

$$\varepsilon = \frac{1}{2} (\nabla s + \nabla^T s), \quad \sigma = \lambda \text{tr}(\varepsilon) \mathbf{I} + 2\mu\varepsilon$$

semi-discretization in space

High order ODE

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Heated rod with delay

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}(x, t) + a_0(x)v(x, t) + a_1(x)v(x, t - 1)$$

$$v(0, t) = u(t) \quad t \geq 0$$

semi-discretization in space

Functional differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_\tau\mathbf{x}(t - \tau) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

- Semi-discretization in space leads to large-scale dynamical systems.
- **Dynamical Structures:** higher order ODE and functional differential equations.
- Other examples: Fading memories, fractional order systems, ...



Introduction

–Structured Linear Dynamical Systems–

Linear Structured Dynamical systems

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))), \quad \mathbf{x}(0) = 0,$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

where

- (generalized) states $\mathbf{x}(t) \in \mathbb{R}^n$ (invertible $\mathbf{E} \in \mathbb{R}^{n \times n}$), \mathbf{f} is linear.
- inputs (controls) $\mathbf{u}(t) \in \mathbb{R}^m$,
- outputs (measurements, quantity of interest) $\mathbf{y}(t) \in \mathbb{R}^q$. In this talk, $m = q = 1$ (SISO).

Linear Structured Dynamical systems

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))), \quad \mathbf{x}(0) = 0,$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

where

- (generalized) states $\mathbf{x}(t) \in \mathbb{R}^n$ (invertible $\mathbf{E} \in \mathbb{R}^{n \times n}$), \mathbf{f} is linear.
- inputs (controls) $\mathbf{u}(t) \in \mathbb{R}^m$,
- outputs (measurements, quantity of interest) $\mathbf{y}(t) \in \mathbb{R}^q$. In this talk, $m = q = 1$ (SISO).

System Classes

Classical linear systems: $\mathbf{f}(\mathbf{x}) := \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$

Delay systems: $\mathbf{f}(\mathbf{x}) := \mathbf{A}\mathbf{x}(t) + \mathbf{A}_\tau\mathbf{x}(t - \tau) + \mathbf{B}\mathbf{u}(t),$

Second-order system $\mathbf{f}(\mathbf{x}) := \mathbf{A}\mathbf{x}(t) + \mathbf{A}_1 \int_0^t \mathbf{x}(\tau)d\tau + \int_0^t \mathbf{B}\mathbf{u}(\tau)\tau, \dots,$

Integro-differential systems: $\mathbf{f}(\mathbf{x}) := \int_0^t \mu(ds)\mathbf{x}(t - s) + \mathbf{B}\mathbf{u}(t),$



Introduction

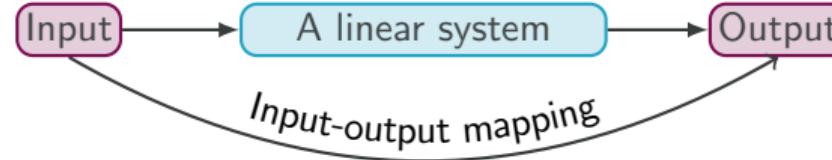
–Structured Linear Dynamical Systems–





Introduction

–Structured Linear Dynamical Systems–

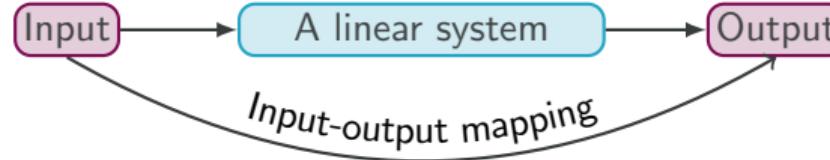


Input-output representation or transfer function



Introduction

–Structured Linear Dynamical Systems–



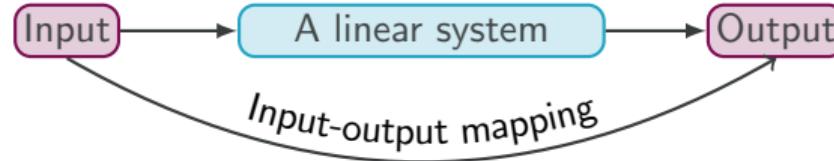
Input-output representation or transfer function

Frequency domain representation of the system ([Laplace transform](#))

$$\mathbf{x}(t) \mapsto \mathbf{X}(s), \quad \dot{\mathbf{x}}(t) \mapsto s\mathbf{X}(s).$$

Linear System

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \quad \mathbf{x}(0) = 0. \end{aligned}$$



Input-output representation or transfer function

Frequency domain representation of the system ([Laplace transform](#))

$$\mathbf{x}(t) \mapsto \mathbf{X}(s), \quad \dot{\mathbf{x}}(t) \mapsto s\mathbf{X}(s).$$

Linear System

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \quad \mathbf{x}(0) = 0. \end{aligned}$$

As a result, we have

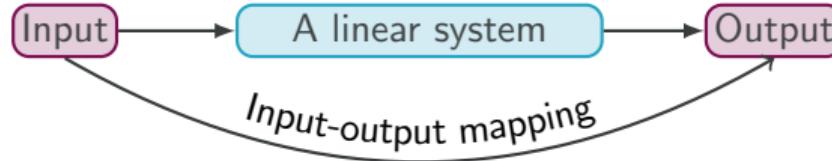
$$s\mathbf{E}\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s), \quad \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s),$$

yielding I/O mapping as

$$\mathbf{Y}(s) = \underbrace{\left(\mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1} \mathbf{B} \right)}_{=: \mathbf{G}(s)} \mathbf{U}(s).$$

Introduction

–Structured Linear Dynamical Systems–



Input-output representation or transfer function

Frequency domain representation of the system ([Laplace transform](#))

$$\mathbf{x}(t) \mapsto \mathbf{X}(s), \quad \dot{\mathbf{x}}(t) \mapsto s\mathbf{X}(s).$$

As a result, we have

$$sE\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s), \quad \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s),$$

yielding I/O mapping as

$$\mathbf{Y}(s) = \underbrace{\left(\mathbf{C}(sE - \mathbf{A})^{-1} \mathbf{B} \right)}_{=: \mathbf{G}(s)} \mathbf{U}(s).$$

Linear System

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad \mathbf{x}(0) = 0.$$

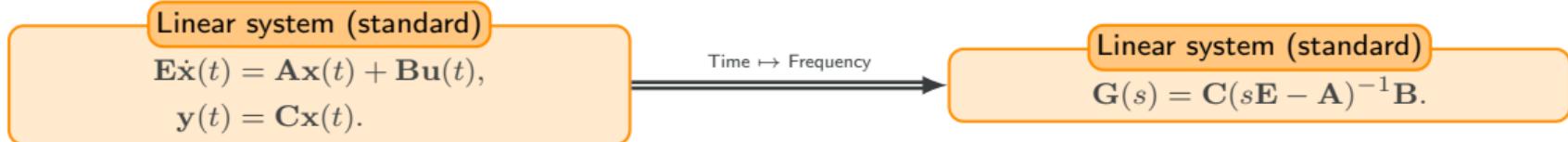
Input-Output Mapping

$$\mathbf{G}(s) := \mathbf{C}(sE - \mathbf{A})^{-1} \mathbf{B}.$$

also known as
transfer function

Introduction

–Structured Linear Dynamical Systems–





CSC

Introduction

–Structured Linear Dynamical Systems–

Linear system (standard)

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{aligned}$$

Time \mapsto Frequency

Linear system (standard)

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}.$$

Second-order system

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) &= \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{aligned}$$

Time \mapsto Frequency

Second-order system

$$\mathbf{G}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{B}.$$



CSC

Introduction

–Structured Linear Dynamical Systems–

Linear system (standard)

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{aligned}$$

Time \mapsto Frequency

Linear system (standard)

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}.$$

Second-order system

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) &= \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{aligned}$$

Time \mapsto Frequency

Second-order system

$$\mathbf{G}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{B}.$$

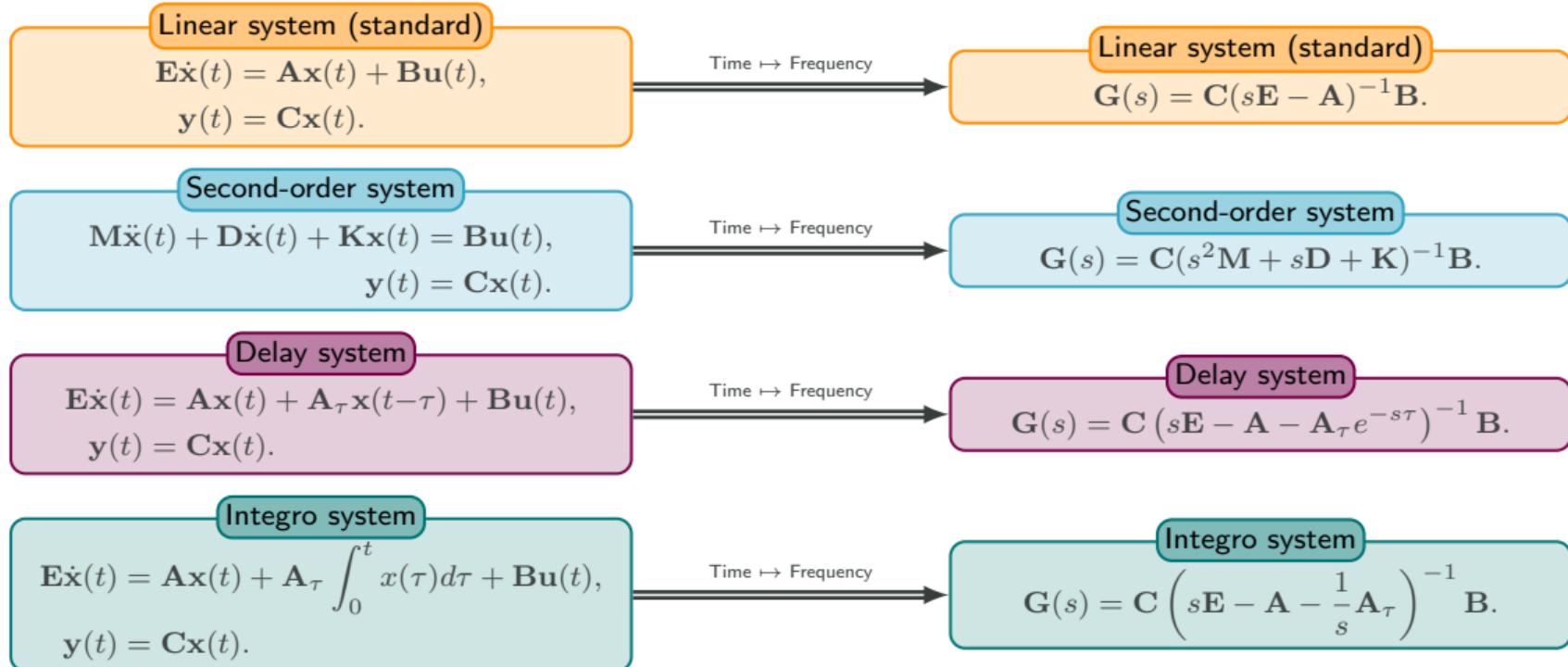
Delay system

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{A}_\tau\mathbf{x}(t-\tau) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{aligned}$$

Time \mapsto Frequency

Delay system

$$\mathbf{G}(s) = \mathbf{C}(\mathbf{s}\mathbf{E} - \mathbf{A} - \mathbf{A}_\tau e^{-s\tau})^{-1}\mathbf{B}.$$





Problem Formulation

Problem Formulation

Approximate the **transfer function** of an n -dimensional system,

$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s),$$

Problem Formulation

Problem Formulation

Approximate the **transfer function** of an n -dimensional system,

$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s),$$

by the transfer function of a system

$$\hat{\mathbf{H}}(s) = \hat{\mathcal{C}}(s)\hat{\mathcal{K}}(s)^{-1}\hat{\mathcal{B}}(s),$$



csc

Problem Formulation

Problem Formulation

Approximate the **transfer function** of an n -dimensional system,

$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s),$$

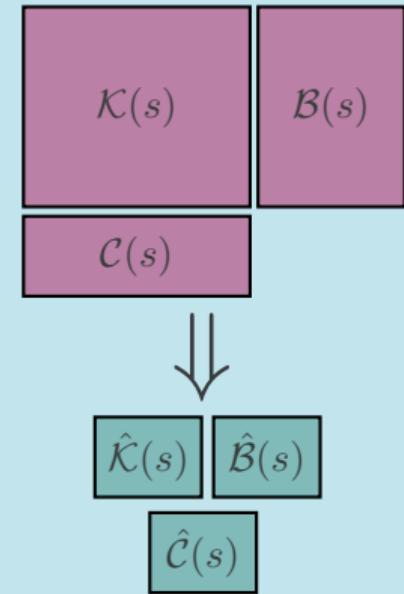
by the transfer function of a system

$$\hat{\mathbf{H}}(s) = \hat{\mathcal{C}}(s)\hat{\mathcal{K}}(s)^{-1}\hat{\mathcal{B}}(s),$$

of **order $r \ll n$** , such that

$$\|\mathbf{H}(s) - \hat{\mathbf{H}}(s)\| < \text{tolerance} \quad \forall s.$$

⇒ Optimization problem: $\min_{\text{order}(\hat{\mathbf{H}}) \leq r} \|\mathbf{H} - \hat{\mathbf{H}}\|.$



Structured system

- $\mathcal{C}(s) = \sum_{i=1}^k \alpha_i(s) \mathbf{C}_i \in \mathbb{R}^{q \times n}$,
 - $\mathcal{K}(s) = \sum_{i=1}^l \beta_i(s) \mathbf{A}_i \in \mathbb{R}^{n \times n}$,
 - $\mathcal{B}(s) = \sum_{i=1}^m \gamma_i(s) \mathbf{B}_i \in \mathbb{R}^{n \times m}$,
 - $\alpha_i(s), \beta_i(s)$ and $\gamma_i(s)$ are meromorphic functions (dynamical structures).
- $\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s)$,

Structured system

- $\mathcal{C}(s) = \sum_{i=1}^k \alpha_i(s) \mathbf{C}_i \in \mathbb{R}^{q \times n}$,
- $\mathcal{K}(s) = \sum_{i=1}^l \beta_i(s) \mathbf{A}_i \in \mathbb{R}^{n \times n}$,
- $\mathcal{B}(s) = \sum_{i=1}^m \gamma_i(s) \mathbf{B}_i \in \mathbb{R}^{n \times m}$,
- $\alpha_i(s), \beta_i(s)$ and $\gamma_i(s)$ are meromorphic functions (dynamical structures).

Structured reduced system

- $\hat{\mathcal{C}}(s) = \sum_{i=1}^k \alpha_i(s) \hat{\mathbf{C}}_i \in \mathbb{R}^{q \times r}$,
- $\hat{\mathcal{K}}(s) = \sum_{i=1}^g \beta_i(s) \hat{\mathbf{A}}_i \in \mathbb{R}^{r \times r}$,
- $\hat{\mathcal{B}}(s) = \sum_{i=1}^m \gamma_i(s) \hat{\mathbf{B}}_i \in \mathbb{R}^{r \times m}$.

- ⇝ Hence, preserve meromorphic functions, and order $r \ll n$.
- How to construct reduced systems, satisfying the desired goals, i.e., $\|\mathbf{H}(s) - \hat{\mathbf{H}}(s)\| \leq \text{tol}$.

Petrov-Galerkin-type projection

For given projection matrices $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}$, leading to

Petrov-Galerkin-type projection

For given projection matrices $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}$, leading to

$$\begin{aligned}\hat{\mathcal{C}}(s) &= \alpha_1(s)\mathbf{C}_1\mathbf{V} + \alpha_2(s)\mathbf{C}_2\mathbf{V} + \cdots + \alpha_k(s)\mathbf{C}_k\mathbf{V}, \\ &= \alpha_1(s)\hat{\mathbf{C}}_1 + \alpha_2(s)\hat{\mathbf{C}}_2 + \cdots + \alpha_k(s)\hat{\mathbf{C}}_k,\end{aligned}$$

$$\begin{aligned}\hat{\mathcal{K}}(s) &= \beta_1(s)\mathbf{W}^T\mathbf{A}_1\mathbf{V} + \cdots + \beta_g(s)\mathbf{W}^T\mathbf{A}_g\mathbf{V}, \\ &= +\beta_1(s)\hat{\mathbf{A}}_1 + \cdots + \beta_g(s)\hat{\mathbf{A}}_g,\end{aligned}$$

$$\begin{aligned}\hat{\mathcal{B}}(s) &= \gamma_1(s)\mathbf{W}^T\mathbf{B}_1 + \gamma_2(s)\mathbf{W}^T\mathbf{B}_2 + \cdots + \gamma_m(s)\mathbf{W}^T\mathbf{B}_m, \\ &= \gamma_1(s)\hat{\mathbf{B}}_1 + \gamma_2(s)\hat{\mathbf{B}}_2 + \cdots + \gamma_m(s)\hat{\mathbf{B}}_m,\end{aligned}$$

Petrov-Galerkin-type projection

For given projection matrices $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}$, leading to

$$\begin{aligned}\hat{\mathcal{C}}(s) &= \alpha_1(s)\mathbf{C}_1\mathbf{V} + \alpha_2(s)\mathbf{C}_2\mathbf{V} + \cdots + \alpha_k(s)\mathbf{C}_k\mathbf{V}, \\ &= \alpha_1(s)\hat{\mathbf{C}}_1 + \alpha_2(s)\hat{\mathbf{C}}_2 + \cdots + \alpha_k(s)\hat{\mathbf{C}}_k,\end{aligned}$$

$$\begin{aligned}\hat{\mathcal{K}}(s) &= \beta_1(s)\mathbf{W}^T\mathbf{A}_1\mathbf{V} + \cdots + \beta_g(s)\mathbf{W}^T\mathbf{A}_g\mathbf{V}, \\ &= +\beta_1(s)\hat{\mathbf{A}}_1 + \cdots + \beta_g(s)\hat{\mathbf{A}}_g,\end{aligned}$$

$$\begin{aligned}\hat{\mathcal{B}}(s) &= \gamma_1(s)\mathbf{W}^T\mathbf{B}_1 + \gamma_2(s)\mathbf{W}^T\mathbf{B}_2 + \cdots + \gamma_m(s)\mathbf{W}^T\mathbf{B}_m, \\ &= \gamma_1(s)\hat{\mathbf{B}}_1 + \gamma_2(s)\hat{\mathbf{B}}_2 + \cdots + \gamma_m(s)\hat{\mathbf{B}}_m,\end{aligned}$$

~~~ Choice of the projection matrices?

## Common existing approaches

## 1. Interpolating reduced-order systems

[BEATTIE/GUGERCIN '09]

$$\mathbf{H}(\sigma_i) = \hat{\mathbf{H}}(\sigma_i), \quad \text{for } i = 1, \dots, 2r$$

- How to choose  $\sigma_i$ ??

## Common existing approaches

### 1. Interpolating reduced-order systems

[BEATTIE/GUGERCIN '09]

$$\mathbf{H}(\sigma_i) = \hat{\mathbf{H}}(\sigma_i), \quad \text{for } i = 1, \dots, 2r$$

- How to choose  $\sigma_i$ ??

### 2. Reduced-order modeling via balancing truncation

[BREITEN '16]

- aims at removing the subspaces those are less important for the dynamics
- Expensive to solve Lyapunov equation



csc

# Some Literature

## Common existing approaches

### 1. Interpolating reduced-order systems

[BEATTIE/GUGERCIN '09]

$$\mathbf{H}(\sigma_i) = \hat{\mathbf{H}}(\sigma_i), \quad \text{for } i = 1, \dots, 2r$$

- How to choose  $\sigma_i$ ??

### 2. Reduced-order modeling via balancing truncation

[BREITEN '16]

- aims at removing the subspaces those are less important for the dynamics
- Expensive to solve Lyapunov equation

### 3. Data-driven structured realization (non-intrusive way)

[SCHULZE ET. AL '18]

- Required expert knowledge and not straightforward to implement.



csc

# Some Literature

## Common existing approaches

### 1. Interpolating reduced-order systems

[BEATTIE/GUGERCIN '09]

$$\mathbf{H}(\sigma_i) = \hat{\mathbf{H}}(\sigma_i), \quad \text{for } i = 1, \dots, 2r$$

- How to choose  $\sigma_i$ ??

### 2. Reduced-order modeling via balancing truncation

[BREITEN '16]

- aims at removing the subspaces those are less important for the dynamics
- Expensive to solve Lyapunov equation

### 3. Data-driven structured realization (non-intrusive way)

[SCHULZE ET. AL '18]

- Required expert knowledge and not straightforward to implement.

### An $n$ -dimensional linear system

$$\Sigma := \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{cases} \quad \Rightarrow \quad \begin{array}{l} \text{Transfer function} \\ \mathbf{H}(s) := \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}. \end{array}$$

### An $n$ -dimensional linear system

$$\Sigma := \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{cases} \quad \Rightarrow \quad \text{Transfer function}$$

$$\mathbf{H}(s) := \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}.$$

### Theorem (simplified)

[VILLEMAGNE/SKELTON 1987, GRIMME 1997]

If

$$\text{range}(\mathbf{V}) \supseteq \text{span}((\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}, \dots, (\sigma_r\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}),$$

$$\text{range}(\mathbf{W}) \supseteq \text{span}((\mu_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{C}, \dots, (\mu_r\mathbf{I} - \mathbf{A})^{-T}\mathbf{C}^T),$$



# First-order systems

## -Interpolation-based MOR-

An  $n$ -dimensional linear system

$$\Sigma := \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{cases} \quad \Rightarrow \quad \text{Transfer function}$$
$$\mathbf{H}(s) := \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}.$$

Theorem (simplified)

[VILLEMAGNE/SKELTON 1987, GRIMME 1997]

If

$$\text{range}(\mathbf{V}) \supseteq \text{span}((\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}, \dots, (\sigma_r\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}),$$

$$\text{range}(\mathbf{W}) \supseteq \text{span}((\mu_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{C}, \dots, (\mu_r\mathbf{I} - \mathbf{A})^{-T}\mathbf{C}^T),$$

then

$$\boxed{\mathbf{H}(s) = \hat{\mathbf{H}}(s), \quad s \in \{\sigma_1, \sigma_r, \mu_1, \dots, \mu_r\}.}$$



# First-order systems

–Reachable and observable subspaces–

An  $n$ -dimensional linear system

$$\Sigma := \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{cases} \quad \Rightarrow$$

- Input to state map:  $\mathbf{x}(t) = \int_0^t e^{A\sigma} \mathbf{B}\mathbf{u}(t - \sigma) d\sigma$
- State to output map:  $\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t} \mathbf{x}_0$ .



# First-order systems

–Reachable and observable subspaces–

An  $n$ -dimensional linear system

$$\Sigma := \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{cases} \quad \Rightarrow$$

- Input to state map:  $\mathbf{x}(t) = \int_0^t e^{A\sigma} \mathbf{B}\mathbf{u}(t - \sigma) d\sigma$
- State to output map:  $\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0.$

Reachable and observable subspaces

The **reachable subspace**  $\mathcal{R}$  and the **observable subspace**  $\mathcal{O}$  are the **smallest subspaces** of  $\mathbb{C}^n$  such that

$$e^{\mathbf{A}t} \mathbf{B} \in \mathcal{R} \quad \text{and} \quad e^{\mathbf{A}^T t} \mathbf{C}^T \in \mathcal{O} \quad \text{for every } t \geq 0.$$



# First-order systems

–Reachable and observable subspaces–

An  $n$ -dimensional linear system

$$\Sigma := \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{cases} \quad \Rightarrow$$

- Input to state map:  $\mathbf{x}(t) = \int_0^t e^{A\sigma} \mathbf{B}\mathbf{u}(t - \sigma) d\sigma$
- State to output map:  $\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0.$

Reachable and observable subspaces

The **reachable subspace**  $\mathcal{R}$  and the **observable subspace**  $\mathcal{O}$  are the **smallest subspaces** of  $\mathbb{C}^n$  such that

$$e^{\mathbf{A}t} \mathbf{B} \in \mathcal{R} \quad \text{and} \quad e^{\mathbf{A}^T t} \mathbf{C}^T \in \mathcal{O} \quad \text{for every } t \geq 0.$$

or, in the frequency domain,

$$(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \in \mathcal{R} \quad \text{and} \quad (s\mathbf{I} - \mathbf{A})^{-T} \mathbf{C}^T \in \mathcal{O} \quad \text{for every } s \in i\mathbb{R}.$$

### An $n$ -dimensional linear system

$$\Sigma := \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t). \end{cases} \quad \Rightarrow$$

- Input to state map:  $\mathbf{x}(t) = \int_0^t e^{A\sigma} \mathbf{B}\mathbf{u}(t - \sigma) d\sigma$
- State to output map:  $\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t} \mathbf{x}_0.$

### Reachable and observable subspaces

The **reachable subspace**  $\mathcal{R}$  and the **observable subspace**  $\mathcal{O}$  are the **smallest subspaces** of  $\mathbb{C}^n$  such that

$$e^{\mathbf{A}t} \mathbf{B} \in \mathcal{R} \quad \text{and} \quad e^{\mathbf{A}^T t} \mathbf{C}^T \in \mathcal{O} \quad \text{for every } t \geq 0.$$

or, in the frequency domain,

$$(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \in \mathcal{R} \quad \text{and} \quad (s\mathbf{I} - \mathbf{A})^{-T} \mathbf{C}^T \in \mathcal{O} \quad \text{for every } s \in i\mathbb{R}.$$

Moreover,  $\mathcal{R}^\perp$  and  $\mathcal{O}^\perp$  are respectively, the **unreachable** and **unobservable subspaces**.



# First-order systems

## –Reachable and observable subspaces–

A classical result in system theory: the **unreachable** ( $\mathcal{R}^\perp$ ) or **unobservable** states ( $\mathcal{O}^\perp$ ) can be removed from the dynamics, without changing the transfer function.



CSC

# First-order systems

## –Reachable and observable subspaces–

A classical result in system theory: the **unreachable** ( $\mathcal{R}^\perp$ ) or **unobservable** states ( $\mathcal{O}^\perp$ ) can be removed from the dynamics, without changing the transfer function.

### Characterization subspaces

- **Krylov subspaces** (R. Kalman):

$$\mathcal{R} = \text{range} \left( [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] \right), \text{ and}$$

$$\mathcal{O} = \text{range} \left( [C^T \quad A^T C^T \quad (A^2)^T C^T \quad \dots \quad (A^{n-1})^T C^T] \right),$$

A classical result in system theory: the **unreachable** ( $\mathcal{R}^\perp$ ) or **unobservable** states ( $\mathcal{O}^\perp$ ) can be removed from the dynamics, without changing the transfer function.

### Characterization subspaces

- **Krylov subspaces** (R. Kalman):

$$\mathcal{R} = \text{range} \left( [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}] \right), \text{ and}$$

$$\mathcal{O} = \text{range} \left( [\mathbf{C}^T \quad \mathbf{A}^T\mathbf{C}^T \quad (\mathbf{A}^2)^T\mathbf{C}^T \quad \dots \quad (\mathbf{A}^{n-1})^T\mathbf{C}^T] \right),$$

- **Rational Krylov subspaces**

[ANDERSON/ANTOULAS 90']

$$\mathcal{R} = \text{range}(\mathbf{R}) = \text{range} \left( [(\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \quad (\sigma_2\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \quad \dots \quad (\sigma_n\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}] \right), \text{ and}$$

$$\mathcal{O} = \text{range}(\mathbf{O}) = \text{range} \left( [(\sigma_1\mathbf{I} - \mathbf{A})^{-T}\mathbf{C}^T \quad (\sigma_2\mathbf{I} - \mathbf{A})^{-T}\mathbf{C}^T \quad \dots \quad (\sigma_n\mathbf{I} - \mathbf{A})^{-T}\mathbf{C}^T] \right)$$

A classical result in system theory: the **unreachable** ( $\mathcal{R}^\perp$ ) or **unobservable** states ( $\mathcal{O}^\perp$ ) can be removed from the dynamics, without changing the transfer function.

### Characterization subspaces

- **Krylov subspaces** (R. Kalman):

$$\mathcal{R} = \text{range} ([\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]), \text{ and}$$

$$\mathcal{O} = \text{range} ([\mathbf{C}^T \quad \mathbf{A}^T\mathbf{C}^T \quad (\mathbf{A}^2)^T\mathbf{C}^T \quad \dots \quad (\mathbf{A}^{n-1})^T\mathbf{C}^T]),$$

- **Rational Krylov subspaces**

[ANDERSON/ANTOULAS 90']

$$\mathcal{R} = \text{range} (\mathbf{R}) = \text{range} ([(\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \quad (\sigma_2\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \quad \dots \quad (\sigma_n\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}]), \text{ and}$$

$$\mathcal{O} = \text{range} (\mathbf{O}) = \text{range} ([(\sigma_1\mathbf{I} - \mathbf{A})^{-T}\mathbf{C}^T \quad (\sigma_2\mathbf{I} - \mathbf{A})^{-T}\mathbf{C}^T \quad \dots \quad (\sigma_n\mathbf{I} - \mathbf{A})^{-T}\mathbf{C}^T])$$

**Notice:**  $\mathbf{R} = \mathbf{V}$  and  $\mathbf{O} = \mathbf{W}$  are the same matrices for interpolation based MOR.

$$\mathbf{R} = [(\sigma_1 \mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \quad (\sigma_2 \mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \quad \dots \quad (\sigma_n \mathbf{I} - \mathbf{A})^{-1} \mathbf{B}], \text{ and}$$

$$\mathbf{O} = [(\sigma_1 \mathbf{I} - \mathbf{A})^{-T} \mathbf{C}^T \quad (\sigma_2 \mathbf{I} - \mathbf{A})^{-T} \mathbf{C}^T \quad \dots \quad (\sigma_n \mathbf{I} - \mathbf{A})^{-T} \mathbf{C}^T]$$

### Minimal order

[ANDERSON/ANTOULAS 90']

$\text{rank}([\mathbf{O}^T \mathbf{R} \quad \mathbf{O}^T \mathbf{A} \mathbf{R}]) = \begin{cases} \text{order of the minimal realization obtained by} \\ \text{removing unreachable and unobservable states} \end{cases}$



CSC

# First-order systems

## -Minimal realization-

$$\mathbf{R} = [(\sigma_1 \mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \quad (\sigma_2 \mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \quad \dots \quad (\sigma_n \mathbf{I} - \mathbf{A})^{-1} \mathbf{B}], \text{ and}$$

$$\mathbf{O} = [(\sigma_1 \mathbf{I} - \mathbf{A})^{-T} \mathbf{C}^T \quad (\sigma_2 \mathbf{I} - \mathbf{A})^{-T} \mathbf{C}^T \quad \dots \quad (\sigma_n \mathbf{I} - \mathbf{A})^{-T} \mathbf{C}^T]$$

### Minimal order

[ANDERSON/ANTOULAS 90']

$$\text{rank}([\mathbf{O}^T \mathbf{R} \quad \mathbf{O}^T \mathbf{A} \mathbf{R}]) = \begin{cases} \text{order of the minimal realization obtained by} \\ \text{removing unreachable and unobservable states} \end{cases}$$

### Construction of minimal or reduced-order approximation

e.g., [MAYO/ANTOULAS '07]

- Matrices  $\mathbf{O}^T \mathbf{R}$  and  $\mathbf{O}^T \mathbf{A} \mathbf{R}$  allow us to find appropriate projection subspaces  $\mathbf{V}$  and  $\mathbf{W}$
- Construction of a minimal system or reduced-order system.



# An Illustrative Example

A demo example

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B},$$

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{C}^T = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$

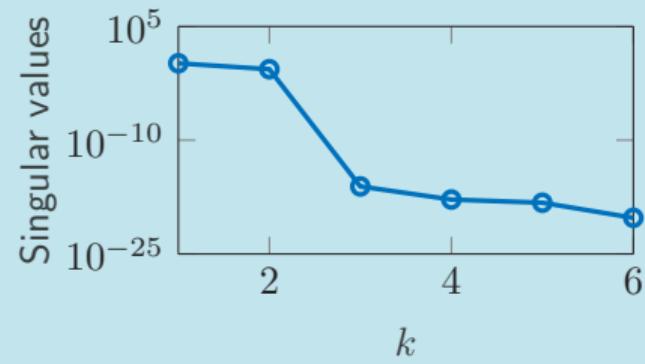
# An Illustrative Example

A demo example

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B},$$

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{C}^T = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$

## Decay of singular values



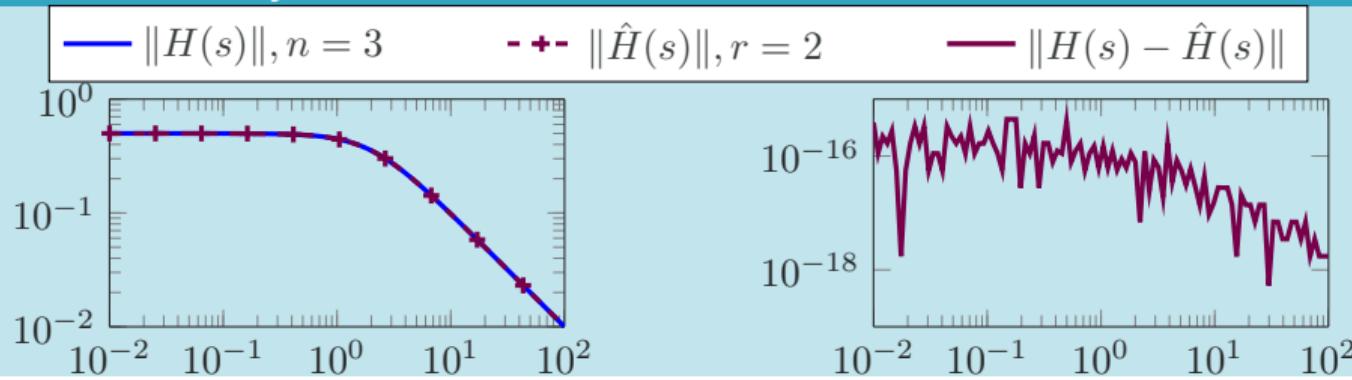
# An Illustrative Example

A demo example

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B},$$

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{C}^T = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$

Construction of a minimal system





# Structured Transfer Function

## –Interpolation–

~~ Can we extend these ideas to structure linear systems?

An  $n$ -dimensional structured linear system

### Transfer function

$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s), .$$



# Structured Transfer Function

## -Interpolation-

~~~ Can we extend these ideas to structure linear systems?

An n -dimensional structured linear system

Transfer function

$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s), .$$

Theorem (simplified)

[BEATTIE/GUGERCIN '09]

If

$$\text{range}(\mathbf{V}) \supseteq \text{span}(\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1), \dots, \mathcal{K}(\sigma_r)^{-1}\mathcal{B}(\sigma_r)),$$

$$\text{range}(\mathbf{W}) \supseteq \text{span}(\mathcal{K}(\mu_1)^{-T}\mathcal{C}(\mu_1)^T, \dots, \mathcal{K}(\mu_r)^{-T}\mathcal{C}(\mu_r)^T),$$



Structured Transfer Function

-Interpolation-

~~ Can we extend these ideas to structure linear systems?

An n -dimensional structured linear system

Transfer function

$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s), .$$

Theorem (simplified)

[BEATTIE/GUGERCIN '09]

If

$$\text{range}(\mathbf{V}) \supseteq \text{span}(\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1), \dots, \mathcal{K}(\sigma_r)^{-1}\mathcal{B}(\sigma_r)),$$

$$\text{range}(\mathbf{W}) \supseteq \text{span}(\mathcal{K}(\mu_1)^{-T}\mathcal{C}(\mu_1)^T, \dots, \mathcal{K}(\mu_r)^{-T}\mathcal{C}(\mu_r)^T),$$

then

$$\boxed{\mathbf{H}(s) = \hat{\mathbf{H}}(s), \quad s \in \{\sigma_1, \sigma_r, \mu_1 \dots, \mu_r\}.}$$

An n -dimensional linear system

Transfer function

$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s), .$$



- Input to state map: $\mathbf{X}(s) = \mathcal{K}(s)^{-1}\mathcal{B}(s)\mathbf{U}(s)$
- State to output map: $\mathbf{Y}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathbf{X}(s).$

An n -dimensional linear system

Transfer function

$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s), .$$



- Input to state map: $\mathbf{X}(s) = \mathcal{K}(s)^{-1}\mathcal{B}(s)\mathbf{U}(s)$
- State to output map: $\mathbf{Y}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathbf{X}(s).$

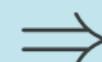
Reachable and observable subspaces for structured systems

The **reachable subspace** \mathcal{R} and the **observable subspace** \mathcal{O} are the smallest subspaces of \mathbb{C}^n such that

$$\mathcal{K}(s)^{-1}\mathcal{B}(s) \in \mathcal{R} \quad \text{and} \quad \mathcal{K}(s)^{-T}\mathcal{C}(s)^T \in \mathcal{O} \quad \text{for every } s \in i\mathbb{R}.$$

An n -dimensional linear system

Transfer function



$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s), .$$

- Input to state map: $\mathbf{X}(s) = \mathcal{K}(s)^{-1}\mathcal{B}(s)\mathbf{U}(s)$
- State to output map: $\mathbf{Y}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathbf{X}(s).$

Reachable and observable subspaces for structured systems

The **reachable subspace** \mathcal{R} and the **observable subspace** \mathcal{O} are the smallest subspaces of \mathbb{C}^n such that

$$\mathcal{K}(s)^{-1}\mathcal{B}(s) \in \mathcal{R} \quad \text{and} \quad \mathcal{K}(s)^{-T}\mathcal{C}(s)^T \in \mathcal{O} \quad \text{for every } s \in i\mathbb{R}.$$

Moreover, \mathcal{R}^\perp and \mathcal{O}^\perp are respectively, the **unreachable** and **unobservable subspaces**.

An n -dimensional linear system

Transfer function



$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s), .$$

- Input to state map: $\mathbf{X}(s) = \mathcal{K}(s)^{-1}\mathcal{B}(s)\mathbf{U}(s)$
- State to output map: $\mathbf{Y}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathbf{X}(s).$

Reachable and observable subspaces for structured systems

The **reachable subspace** \mathcal{R} and the **observable subspace** \mathcal{O} are the smallest subspaces of \mathbb{C}^n such that

$$\mathcal{K}(s)^{-1}\mathcal{B}(s) \in \mathcal{R} \quad \text{and} \quad \mathcal{K}(s)^{-T}\mathcal{C}(s)^T \in \mathcal{O} \quad \text{for every } s \in i\mathbb{R}.$$

Moreover, \mathcal{R}^\perp and \mathcal{O}^\perp are respectively, the **unreachable** and **unobservable subspaces**.

A result in for structured linear systems: the **unreachable** (\mathcal{R}^\perp) or **unobservable** states (\mathcal{O}^\perp) can be removed from the dynamics, without changing the transfer function.

Structured transfer function

Consider an n -dimensional linear system, whose structure transfer function is given by

$$\mathbf{H}(s) := \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s).$$

Characterization Controllable and Observable Subspaces (simplified)

[BENNER/GOYAL/P. '19]

Let

$$\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1) \quad \mathcal{K}(\sigma_2)^{-1}\mathcal{B}(\sigma_2) \quad \dots \quad \mathcal{K}(\sigma_g)^{-1}\mathcal{B}(\sigma_g)],$$

$$\mathbf{O} = [\mathcal{K}(\sigma_1)^{-T}\mathcal{C}(\sigma_1)^T \quad \mathcal{K}(\sigma_2)^{-T}\mathcal{C}(\sigma_2)^T \quad \dots \quad \mathcal{K}(\sigma_g)^{-T}\mathcal{C}(\sigma_g)^T].$$

Structured transfer function

Consider an n -dimensional linear system, whose structure transfer function is given by

$$\mathbf{H}(s) := \mathcal{C}(s)\mathcal{K}(s)^{-1}B(s).$$

Characterization Controllable and Observable Subspaces (simplified)

[BENNER/GOYAL/P. '19]

Let

$$\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1) \quad \mathcal{K}(\sigma_2)^{-1}\mathcal{B}(\sigma_2) \quad \dots \quad \mathcal{K}(\sigma_g)^{-1}\mathcal{B}(\sigma_g)],$$

$$\mathbf{O} = [\mathcal{K}(\sigma_1)^{-T}\mathcal{C}(\sigma_1)^T \quad \mathcal{K}(\sigma_2)^{-T}\mathcal{C}(\sigma_2)^T \quad \dots \quad \mathcal{K}(\sigma_g)^{-T}\mathcal{C}(\sigma_g)^T].$$

Then

■ Reachable subspace:

$$\mathcal{R} = \text{range}(\mathbf{R}).$$

■ Observable subspace:

$$\mathcal{O} = \text{range}(\mathbf{O}).$$

Structured transfer function

Consider an n -dimensional linear system, whose structure transfer function is given by

$$\mathbf{H}(s) := \mathcal{C}(s)\mathcal{K}(s)^{-1}B(s).$$

Characterization Controllable and Observable Subspaces (simplified)

[BENNER/GOYAL/P. '19]

Let

$$\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1) \quad \mathcal{K}(\sigma_2)^{-1}\mathcal{B}(\sigma_2) \quad \dots \quad \mathcal{K}(\sigma_g)^{-1}\mathcal{B}(\sigma_g)],$$

$$\mathbf{O} = [\mathcal{K}(\sigma_1)^{-T}\mathcal{C}(\sigma_1)^T \quad \mathcal{K}(\sigma_2)^{-T}\mathcal{C}(\sigma_2)^T \quad \dots \quad \mathcal{K}(\sigma_g)^{-T}\mathcal{C}(\sigma_g)^T].$$

Then

- Reachable subspace: $\mathcal{R} = \text{range}(\mathbf{R})$.
- Observable subspace: $\mathcal{O} = \text{range}(\mathbf{O})$.

Notice: $\mathbf{R} = \mathbf{V}$ and $\mathbf{O} = \mathbf{W}$ are the same matrices for interpolation based MOR.

Structured transfer function

Consider an n-dimensional linear system, whose structure transfer function is given by

$$\mathbf{H}(s) := \mathcal{C}(s)\mathcal{K}(s)^{-1}B(s), \quad \text{with} \quad \mathcal{K}(s) = \sum_{i=1}^l \beta_i(s)\mathbf{A}_i,$$

and let

$$\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1) \quad \mathcal{K}(\sigma_2)^{-1}\mathcal{B}(\sigma_2) \quad \dots \quad \mathcal{K}(\sigma_N)^{-1}\mathcal{B}(\sigma_N)],$$

$$\mathbf{O} = [\mathcal{K}(\sigma_1)^{-T}\mathcal{C}(\sigma_1)^T \quad \mathcal{K}(\sigma_2)^{-T}\mathcal{C}(\sigma_2)^T \quad \dots \quad \mathcal{K}(\sigma_N)^{-T}\mathcal{C}(\sigma_N)^T].$$

such that $\mathcal{R} = \text{range}(\mathbf{R})$ and $\mathcal{O} = \text{range}(\mathbf{O})$.

Structured transfer function

Consider an n-dimensional linear system, whose structure transfer function is given by

$$\mathbf{H}(s) := \mathcal{C}(s)\mathcal{K}(s)^{-1}B(s), \quad \text{with} \quad \mathcal{K}(s) = \sum_{i=1}^l \beta_i(s)\mathbf{A}_i,$$

and let

$$\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1) \quad \mathcal{K}(\sigma_2)^{-1}\mathcal{B}(\sigma_2) \quad \dots \quad \mathcal{K}(\sigma_N)^{-1}\mathcal{B}(\sigma_N)],$$

$$\mathbf{O} = [\mathcal{K}(\sigma_1)^{-T}\mathcal{C}(\sigma_1)^T \quad \mathcal{K}(\sigma_2)^{-T}\mathcal{C}(\sigma_2)^T \quad \dots \quad \mathcal{K}(\sigma_N)^{-T}\mathcal{C}(\sigma_N)^T].$$

such that $\mathcal{R} = \text{range}(\mathbf{R})$ and $\mathcal{O} = \text{range}(\mathbf{O})$.

Minimal order (simplified)

[BENNER/GOYAL/P. '19]

$$\text{rank}([\mathbf{O}^T \mathbf{A}_1 \mathbf{R} \quad \dots \quad \mathbf{O}^T \mathbf{A}_l \mathbf{R}]) = \begin{cases} \text{order of the minimal realization obtained by} \\ \text{removing unreachable and unobservable states} \end{cases}$$

We propose a method enabling to identify simultaneously the states that are unreachable and unobservable. Assume

$$\text{rank} \left(\begin{bmatrix} \mathbf{O}^T \mathbf{A}_1 \mathbf{R} & \dots & \mathbf{O}^T \mathbf{A}_l \mathbf{R} \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} \mathbf{O}^T \mathbf{A}_1 \mathbf{R} \\ \vdots \\ \mathbf{O}^T \mathbf{A}_l \mathbf{R} \end{bmatrix} \right) = r,$$

with $\text{range}(\mathbf{R}) = \mathcal{R}$ and $\text{range}(\mathbf{O}) = \mathcal{O}$.

We propose a method enabling to identify simultaneously the states that are unreachable and unobservable. Assume

$$\text{rank} \left(\begin{bmatrix} \mathbf{O}^T \mathbf{A}_1 \mathbf{R} & \dots & \mathbf{O}^T \mathbf{A}_l \mathbf{R} \end{bmatrix} \right) = \text{rank} \begin{pmatrix} \begin{bmatrix} \mathbf{O}^T \mathbf{A}_1 \mathbf{R} \\ \vdots \\ \mathbf{O}^T \mathbf{A}_l \mathbf{R} \end{bmatrix} \end{pmatrix} = r,$$

with $\text{range}(\mathbf{R}) = \mathcal{R}$ and $\text{range}(\mathbf{O}) = \mathcal{O}$. Then, we consider the compact SVDs

$$\begin{bmatrix} \mathbf{O}^T \mathbf{A}_1 \mathbf{R} & \dots & \mathbf{O}^T \mathbf{A}_l \mathbf{R} \end{bmatrix} = \mathbf{W}_1 \Sigma_l \tilde{\mathbf{V}}^T \quad \text{and} \quad \begin{bmatrix} \mathbf{O}^T \mathbf{A}_1 \mathbf{R} \\ \vdots \\ \mathbf{O}^T \mathbf{A}_l \mathbf{R} \end{bmatrix} = \tilde{\mathbf{W}} \Sigma_r \mathbf{V}_1^T.$$

Let $\mathbf{W} := \mathbf{O} \mathbf{W}_1$ and $\mathbf{V} := \mathbf{R} \mathbf{V}_1$ be two projection matrices and let us consider the lower-order realization $\hat{\mathcal{C}}(s)\hat{\mathcal{K}}(s)^{-1}\hat{\mathcal{B}}(s)$ constructed by Petrov-Galerkin projection.

Petrov-Galerkin projections:

$$\mathbf{W} := \mathbf{O}\mathbf{W}_1 \text{ and } \mathbf{V} := \mathbf{R}\mathbf{V}_1$$

Theorem

[BENNER/GOYAL/P. '19]

The lower-order system $\hat{\mathcal{C}}(s)\hat{\mathcal{K}}(s)^{-1}\hat{\mathcal{B}}(s)$ of order r , obtained by Petrov-Galerkin projection with \mathbf{V} and \mathbf{W} , realizes the original transfer function, i.e.,

$$\hat{\mathcal{C}}(s)\hat{\mathcal{K}}(s)^{-1}\hat{\mathcal{B}}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s)$$

for every $s \in i\mathbb{R}$.

Petrov-Galerkin projections:

$$\mathbf{W} := \mathbf{O}\mathbf{W}_1 \text{ and } \mathbf{V} := \mathbf{R}\mathbf{V}_1$$

Theorem

[BENNER/Goyal/P. '19]

The lower-order system $\hat{\mathcal{C}}(s)\hat{\mathcal{K}}(s)^{-1}\hat{\mathcal{B}}(s)$ of order r , obtained by Petrov-Galerkin projection with \mathbf{V} and \mathbf{W} , realizes the original transfer function, i.e.,

$$\hat{\mathcal{C}}(s)\hat{\mathcal{K}}(s)^{-1}\hat{\mathcal{B}}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s)$$

for every $s \in i\mathbb{R}$.

Determine dominate reachable and observable subspaces

- The proposed procedure remove uncontrollable and unobservable subspaces simultaneously.

Petrov-Galerkin projections:

$$\mathbf{W} := \mathbf{O}\mathbf{W}_1 \text{ and } \mathbf{V} := \mathbf{R}\mathbf{V}_1$$

Theorem

[BENNER/GOYAL/P. '19]

The lower-order system $\hat{\mathcal{C}}(s)\hat{\mathcal{K}}(s)^{-1}\hat{\mathcal{B}}(s)$ of order r , obtained by Petrov-Galerkin projection with \mathbf{V} and \mathbf{W} , realizes the original transfer function, i.e.,

$$\hat{\mathcal{C}}(s)\hat{\mathcal{K}}(s)^{-1}\hat{\mathcal{B}}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s)$$

for every $s \in i\mathbb{R}$.

Determine dominate reachable and observable subspaces

- The proposed procedure remove uncontrollable and unobservable subspaces simultaneously.
- Neglecting small singular values leads to reduced-order models.

Petrov-Galerkin projections:

$$\mathbf{W} := \mathbf{O}\mathbf{W}_1 \text{ and } \mathbf{V} := \mathbf{R}\mathbf{V}_1$$

Theorem

[BENNER/GOYAL/P. '19]

The lower-order system $\hat{\mathcal{C}}(s)\hat{\mathcal{K}}(s)^{-1}\hat{\mathcal{B}}(s)$ of order r , obtained by Petrov-Galerkin projection with \mathbf{V} and \mathbf{W} , realizes the original transfer function, i.e.,

$$\hat{\mathcal{C}}(s)\hat{\mathcal{K}}(s)^{-1}\hat{\mathcal{B}}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s)$$

for every $s \in i\mathbb{R}$.

Determine dominate reachable and observable subspaces

- The proposed procedure remove uncontrollable and unobservable subspaces simultaneously.
- Neglecting small singular values leads to reduced-order models.
- Like balanced truncation, order the vectors in order of their importance.

Algorithm: Dominant Reachable and Observable Projection (DROP)

1. Take $\sigma_i, \mu_i, i = 1, \dots, \mathcal{N}$.

Algorithm: Dominant Reachable and Observable Projection (DROP)

1. Take $\sigma_i, \mu_i, i = 1, \dots, \mathcal{N}$.
2. Compute $\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1), \dots, \mathcal{K}(\sigma_{\mathcal{N}})^{-1}\mathcal{B}(\sigma_{\mathcal{N}})]$.
3. Compute $\mathbf{O} = [\mathcal{K}(\mu_1)^{-T}\mathcal{C}(\mu_1)^T, \dots, \mathcal{K}(\mu_{\mathcal{N}})^{-T}\mathcal{C}(\mu_{\mathcal{N}})^T]$.

Algorithm: Dominant Reachable and Observable Projection (DROP)

1. Take $\sigma_i, \mu_i, i = 1, \dots, \mathcal{N}$.
2. Compute $\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1), \dots, \mathcal{K}(\sigma_{\mathcal{N}})^{-1}\mathcal{B}(\sigma_{\mathcal{N}})]$.
3. Compute $\mathbf{O} = [\mathcal{K}(\mu_1)^{-T}\mathcal{C}(\mu_1)^T, \dots, \mathcal{K}(\mu_{\mathcal{N}})^{-T}\mathcal{C}(\mu_{\mathcal{N}})^T]$.
4. Determine $\mathbb{L}^{(i)} = \mathbf{O}^T \mathbf{A}_i \mathbf{R}$.

Algorithm: Dominant Reachable and Observable Projection (DROP)

1. Take $\sigma_i, \mu_i, i = 1, \dots, \mathcal{N}$.
2. Compute $\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1), \dots, \mathcal{K}(\sigma_{\mathcal{N}})^{-1}\mathcal{B}(\sigma_{\mathcal{N}})]$.
3. Compute $\mathbf{O} = [\mathcal{K}(\mu_1)^{-T}\mathcal{C}(\mu_1)^T, \dots, \mathcal{K}(\mu_{\mathcal{N}})^{-T}\mathcal{C}(\mu_{\mathcal{N}})^T]$.
4. Determine $\mathbb{L}^{(i)} = \mathbf{O}^T \mathbf{A}_i \mathbf{R}$.
5. Compute singular value decomposition:

$$[\mathbf{Y}_1, \Sigma_1, \mathbf{X}_1] = \text{svd}([\mathbb{L}^{(1)}, \dots, \mathbb{L}^{(l)}]), \quad [\mathbf{Y}_2, \Sigma_2, \mathbf{X}_2] = \text{svd} \left(\begin{bmatrix} \mathbb{L}^{(1)} \\ \vdots \\ \mathbb{L}^{(l)} \end{bmatrix} \right).$$

Algorithm: Dominant Reachable and Observable Projection (DROP)

1. Take $\sigma_i, \mu_i, i = 1, \dots, \mathcal{N}$.
2. Compute $\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1), \dots, \mathcal{K}(\sigma_{\mathcal{N}})^{-1}\mathcal{B}(\sigma_{\mathcal{N}})]$.
3. Compute $\mathbf{O} = [\mathcal{K}(\mu_1)^{-T}\mathcal{C}(\mu_1)^T, \dots, \mathcal{K}(\mu_{\mathcal{N}})^{-T}\mathcal{C}(\mu_{\mathcal{N}})^T]$.
4. Determine $\mathbb{L}^{(i)} = \mathbf{O}^T \mathbf{A}_i \mathbf{R}$.
5. Compute singular value decomposition:

$$[\mathbf{Y}_1, \Sigma_1, \mathbf{X}_1] = \text{svd}([\mathbb{L}^{(1)}, \dots, \mathbb{L}^{(l)}]), \quad [\mathbf{Y}_2, \Sigma_2, \mathbf{X}_2] = \text{svd} \left(\begin{bmatrix} \mathbb{L}^{(1)} \\ \vdots \\ \mathbb{L}^{(l)} \end{bmatrix} \right).$$

6. Determine projection matrices: $\mathbf{V} := \mathbf{R} \mathbf{X}_2(:, 1:r)$, $\mathbf{W} := \mathbf{O} \mathbf{Y}_1(:, 1:r)$.

Algorithm: Dominant Reachable and Observable Projection (DROP)

1. Take $\sigma_i, \mu_i, i = 1, \dots, \mathcal{N}$.
2. Compute $\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1), \dots, \mathcal{K}(\sigma_{\mathcal{N}})^{-1}\mathcal{B}(\sigma_{\mathcal{N}})]$.
3. Compute $\mathbf{O} = [\mathcal{K}(\mu_1)^{-T}\mathcal{C}(\mu_1)^T, \dots, \mathcal{K}(\mu_{\mathcal{N}})^{-T}\mathcal{C}(\mu_{\mathcal{N}})^T]$.
4. Determine $\mathbb{L}^{(i)} = \mathbf{O}^T \mathbf{A}_i \mathbf{R}$.
5. Compute singular value decomposition:

$$[\mathbf{Y}_1, \Sigma_1, \mathbf{X}_1] = \text{svd}([\mathbb{L}^{(1)}, \dots, \mathbb{L}^{(l)}]), \quad [\mathbf{Y}_2, \Sigma_2, \mathbf{X}_2] = \text{svd} \left(\begin{bmatrix} \mathbb{L}^{(1)} \\ \vdots \\ \mathbb{L}^{(l)} \end{bmatrix} \right).$$

6. Determine projection matrices: $\mathbf{V} := \mathbf{R} \mathbf{X}_2(:, 1:r)$, $\mathbf{W} := \mathbf{O} \mathbf{Y}_1(:, 1:r)$.
7. Determine reduced-order system

$$\hat{\mathcal{K}}(s) = \mathbf{W}^T \mathcal{K}(s) \mathbf{V}, \quad \hat{\mathcal{B}}(s) = \mathbf{W}^T \mathcal{B}(s), \quad \hat{\mathcal{C}}(s) = \mathcal{C}(s) \mathbf{V}.$$



csc

Algorithm to Construct Structured ROMs

Algorithm: Dominant Reachable and Observable Projection

1. Take $\sigma_i, \mu_i, i = 1, \dots, \mathcal{N}$.

2. Compute $\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1), \dots, \mathcal{K}(\sigma_{\mathcal{N}})^{-1}\mathcal{B}(\sigma_{\mathcal{N}})]$.

3. Compute $\mathbf{O} = [\mathcal{K}(\mu_1)^{-T}\mathcal{C}(\mu_1)^T, \dots, \mathcal{K}(\mu_{\mathcal{N}})^{-T}\mathcal{C}(\mu_{\mathcal{N}})^T]$.

4. Determine $\mathbb{L}^{(i)} = \mathbf{O}^T \mathbf{A}_i \mathbf{R}$.

5. Compute singular value decomposition:

$$[\mathbf{Y}_1, \Sigma_1, \mathbf{X}_1] = \text{svd}([\mathbb{L}^{(1)}, \dots, \mathbb{L}^{(l)}]), \quad [\mathbf{Y}_2, \Sigma_2, \mathbf{X}_2] = \text{svd}\left(\begin{bmatrix} \mathbb{L}^{(1)} \\ \vdots \\ \mathbb{L}^{(l)} \end{bmatrix}\right).$$

6. Determine projection matrices: $\mathbf{V} := \mathbf{R} \mathbf{X}_2(:, 1:r)$, $\mathbf{W} := \mathbf{O} \mathbf{Y}_1(:, 1:r)$.

7. Determine reduced-order system

$$\hat{\mathcal{K}}(s) = \mathbf{W}^T \mathcal{K}(s) \mathbf{V}, \quad \hat{\mathcal{B}}(s) = \mathbf{W}^T \mathcal{B}(s), \quad \hat{\mathcal{C}}(s) = \mathcal{C}(s) \mathbf{V}.$$

- Can be easily parallelized
- not need to solve all shifted systems
- Make use of Low-rank solvers.



csc

Algorithm to Construct Structured ROMs

Algorithm: Dominant Reachable and Observable Projection

1. Take $\sigma_i, \mu_i, i = 1, \dots, \mathcal{N}$.

- Can be easily parallelized
- not need to solve all shifted systems
- Make use of Low-rank solvers.

2. Compute $\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1), \dots, \mathcal{K}(\sigma_{\mathcal{N}})^{-1}\mathcal{B}(\sigma_{\mathcal{N}})]$.

3. Compute $\mathbf{O} = [\mathcal{K}(\mu_1)^{-T}\mathcal{C}(\mu_1)^T, \dots, \mathcal{K}(\mu_{\mathcal{N}})^{-T}\mathcal{C}(\mu_{\mathcal{N}})^T]$.

4. Determine $\mathbb{L}^{(i)} = \mathbf{O}^T \mathbf{A}_i \mathbf{R}$.

5. Compute singular value decomposition:

$$[\mathbf{Y}_1, \Sigma_1, \mathbf{X}_1] = \text{svd}([\mathbb{L}^{(1)}, \dots, \mathbb{L}^{(l)}]), \quad [\mathbf{Y}_2, \Sigma_2, \mathbf{X}_2] = \text{svd} \left(\begin{bmatrix} \mathbb{L}^{(1)} \\ \vdots \\ \mathbb{L}^{(l)} \end{bmatrix} \right).$$

6. Determine projection matrices: $\mathbf{V} := \mathbf{R} \mathbf{X}_2(:, 1:r)$,

- Efficient variant of SVDs can be applied including randomized SVD.

7. Determine reduced-order system

$$\hat{\mathcal{K}}(s) = \mathbf{W}^T \mathcal{K}(s) \mathbf{V}, \quad \hat{\mathcal{B}}(s) = \mathbf{W}^T \mathcal{B}(s), \quad \hat{\mathcal{C}}(s) = \mathcal{C}(s) \mathbf{V}.$$

- If we take enough points (σ_i) , the matrix

$$\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1) \quad \dots \quad \mathcal{K}(\sigma_N)^{-1}\mathcal{B}(\sigma_N)],$$

encodes the \mathbb{C}^n reachable subspace.

- If we take enough points (σ_i) , the matrix

$$\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1) \quad \dots \quad \mathcal{K}(\sigma_N)^{-1}\mathcal{B}(\sigma_N)],$$

encodes the \mathbb{C}^n reachable subspace.

Notice that \mathbf{R} solves

$$\sum_{i=1}^l \mathbf{A}_i \mathbf{RM}_i = \sum_{i=1}^m \mathbf{B}_i \mathbf{b}_i,$$

where $\mathbf{M}_i = \text{diag}(\beta_i(\sigma_1), \dots, \beta_i(\sigma_N))$ and $\mathbf{b}_i = [\gamma_i(\sigma_1), \dots, \gamma_i(\sigma_N)]$.



csc

Low Rank Solvers for Sylvester Equations

- If we take enough points (σ_i) , the matrix

$$\mathbf{R} = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1) \quad \dots \quad \mathcal{K}(\sigma_N)^{-1}\mathcal{B}(\sigma_N)],$$

encodes the \mathbb{C}^n reachable subspace.

Notice that \mathbf{R} solves

$$\sum_{i=1}^l \mathbf{A}_i \mathbf{RM}_i = \sum_{i=1}^m \mathbf{B}_i \mathbf{b}_i,$$

where $\mathbf{M}_i = \text{diag}(\beta_i(\sigma_1), \dots, \beta_i(\sigma_N))$ and $\mathbf{b}_i = [\gamma_i(\sigma_1), \dots, \gamma_i(\sigma_N)]$.

- It is a generalized Sylvester equation.
- Low-rank solution is suitable.
- Truncated low-rank methods for generalized Sylvester equation.

[KRESSNER, SIRKOVIC 15']



Parametric extension

–Parametric Structured Linear Systems–

We also consider dynamical systems that are linear in state and parameterized.

We also consider dynamical systems that are linear in state and parameterized.

Parametric Butterfly Gyroscope

[MORWIKI, MODIFIED GYROSCOPE]

$$\begin{aligned}\mathbf{M}(d)\ddot{\mathbf{x}}(t) + \mathbf{D}(d, \theta)\dot{\mathbf{x}}(t) + \mathbf{K}(\theta) = \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),\end{aligned}$$

where

$$\mathbf{M}(d) = \mathbf{M}_1 + d\mathbf{M}_2 \in \mathbb{R}^n,$$

$$\mathbf{D}(d, \theta) = \theta (\mathbf{D}_1 + d\mathbf{D}_2),$$

$$\mathbf{K}(d) = \mathbf{T}_1 + \frac{1}{d}\mathbf{T}_2 + d\mathbf{T}_3.$$

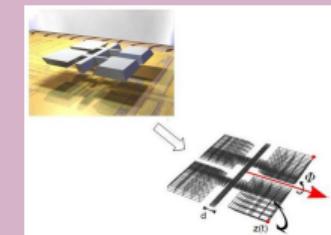


Figure: Semantic gyroscope diagram.

Parameters and frequency range:

$$\theta \in [10^{-5}, 10^{-7}], d \in [1, 2] \quad f \in [0.025, 40].$$

Order of the system: 17,913.

Parametric Problem Formulation

Approximate the **transfer function** of an n -dimensional system,

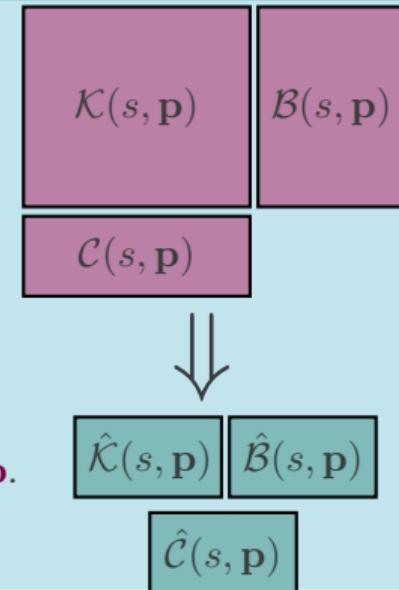
$$\mathbf{H}(s, \mathbf{p}) = \mathcal{C}(s, \mathbf{p})\mathcal{K}(s, \mathbf{p})^{-1}\mathcal{B}(s, \mathbf{p}),$$

by the transfer function of a system

$$\hat{\mathbf{H}}(s, \mathbf{p}) = \hat{\mathcal{C}}(s, \mathbf{p})\hat{\mathcal{K}}(s, \mathbf{p})^{-1}\hat{\mathcal{B}}(s, \mathbf{p}),$$

of **order $r \ll n$** , such that

$$\|\mathbf{H}(s, \mathbf{p}) - \hat{\mathbf{H}}(s, \mathbf{p})\| < \text{tolerance} \quad \forall s \text{ and } \forall \mathbf{p}.$$



Parametric structured linear system

- $\mathbf{H}(s) = \mathcal{C}(s, \mathbf{p})\mathcal{K}(s, \mathbf{p})^{-1}\mathcal{B}(s, \mathbf{p}),$
- $\mathcal{C}(s, \mathbf{p}) = \sum_{i=1}^k \alpha_i(s, \mathbf{p})\mathbf{C}_i \in \mathbb{R}^{q \times n},$
 - $\mathcal{K}(s, \mathbf{p}) = \sum_{i=1}^l \beta_i(s, \mathbf{p})\mathbf{A}_i \in \mathbb{R}^{n \times n},$
 - $\mathcal{B}(s, \mathbf{p}) = \sum_{i=1}^m \gamma_i(s, \mathbf{p})\mathbf{B}_i \in \mathbb{R}^{n \times m},$

Parametric structured linear system

- $\mathcal{C}(s, \mathbf{p}) = \sum_{i=1}^k \alpha_i(s, \mathbf{p}) \mathbf{C}_i \in \mathbb{R}^{q \times n},$

- $\mathcal{K}(s, \mathbf{p}) = \sum_{i=1}^l \beta_i(s, \mathbf{p}) \mathbf{A}_i \in \mathbb{R}^{n \times n},$

- $\mathcal{B}(s, \mathbf{p}) = \sum_{i=1}^m \gamma_i(s, \mathbf{p}) \mathbf{B}_i \in \mathbb{R}^{n \times m},$

Reachable and observable subspaces for parametric structured systems

The **reachable subspace** \mathcal{R} and the **observable subspace** \mathcal{O} are the smallest subspaces of \mathbb{C}^n such that

$$\mathcal{K}(s, \mathbf{p})^{-1} \mathcal{B}(s, \mathbf{p}) \in \mathcal{R} \quad \text{and} \quad \mathcal{K}(s, \mathbf{p})^{-T} \mathcal{C}(s, \mathbf{p})^T \in \mathcal{O} \quad \text{for every } s \in i\mathbb{R} \text{ and } \mathbf{p} \in \Omega.$$

Parametric structured linear system

- $\mathcal{C}(s, \mathbf{p}) = \sum_{i=1}^k \alpha_i(s, \mathbf{p}) \mathbf{C}_i \in \mathbb{R}^{q \times n},$

- $\mathcal{K}(s, \mathbf{p}) = \sum_{i=1}^l \beta_i(s, \mathbf{p}) \mathbf{A}_i \in \mathbb{R}^{n \times n},$

- $\mathcal{B}(s, \mathbf{p}) = \sum_{i=1}^m \gamma_i(s, \mathbf{p}) \mathbf{B}_i \in \mathbb{R}^{n \times m},$

Reachable and observable subspaces for parametric structured systems

The **reachable subspace** \mathcal{R} and the **observable subspace** \mathcal{O} are the smallest subspaces of \mathbb{C}^n such that

$$\mathcal{K}(s, \mathbf{p})^{-1} \mathcal{B}(s, \mathbf{p}) \in \mathcal{R} \quad \text{and} \quad \mathcal{K}(s, \mathbf{p})^{-T} \mathcal{C}(s, \mathbf{p})^T \in \mathcal{O} \quad \text{for every } s \in i\mathbb{R} \text{ and } \mathbf{p} \in \Omega.$$

$$\mathbf{R} = [\mathcal{K}(\sigma_1, \mathbf{p}_1)^{-1} \mathcal{B}(\sigma_1, \mathbf{p}_1) \quad \mathcal{K}(\sigma_2, \mathbf{p}_2)^{-1} \mathcal{B}(\sigma_2, \mathbf{p}_2) \quad \dots \quad \mathcal{K}(\sigma_g, \mathbf{p}_g)^{-1} \mathcal{B}(\sigma_g, \mathbf{p}_g)],$$

$$\mathbf{O} = [\mathcal{K}(\sigma_1, \mathbf{p}_1)^{-T} \mathcal{C}(\sigma_1, \mathbf{p}_1)^T \quad \mathcal{K}(\sigma_2, \mathbf{p}_2)^{-T} \mathcal{C}(\sigma_2, \mathbf{p}_2)^T \quad \dots \quad \mathcal{K}(\sigma_g, \mathbf{p}_g)^{-T} \mathcal{C}(\sigma_g, \mathbf{p}_g)^T].$$

Then, if we have enough interpolation points, $\mathcal{R} = \text{range}(\mathbf{R})$ and $\mathcal{O} = \text{range}(\mathbf{O})$.

Structured transfer function

Consider an n -dimensional linear system, whose structure transfer function is given by

$$\mathbf{H}(s, \mathbf{p}) := \mathcal{C}(s, \mathbf{p}) \mathcal{K}(s, \mathbf{p})^{-1} B(s, \mathbf{p}), \quad \text{with} \quad \mathcal{K}(s, \mathbf{p}) = \sum_{i=1}^l \beta_i(s, \mathbf{p}) \mathbf{A}_i,$$

Structured transfer function

Consider an n -dimensional linear system, whose structure transfer function is given by

$$\mathbf{H}(s, \mathbf{p}) := \mathcal{C}(s, \mathbf{p}) \mathcal{K}(s, \mathbf{p})^{-1} B(s, \mathbf{p}), \quad \text{with} \quad \mathcal{K}(s, \mathbf{p}) = \sum_{i=1}^l \beta_i(s, \mathbf{p}) \mathbf{A}_i,$$

Minimal order (simplified)

[BENNER/GOYAL/P. '19]

$$\text{rank} \left([\mathbf{O}^T \mathbf{A}_1 \mathbf{R} \quad \dots \quad \mathbf{O}^T \mathbf{A}_l \mathbf{R}] \right) = \begin{cases} \text{order of the minimal realization obtained by} \\ \text{removing unreachable and unobservable states} \end{cases}$$

Structured transfer function

Consider an n -dimensional linear system, whose structure transfer function is given by

$$\mathbf{H}(s, \mathbf{p}) := \mathcal{C}(s, \mathbf{p}) \mathcal{K}(s, \mathbf{p})^{-1} B(s, \mathbf{p}), \quad \text{with} \quad \mathcal{K}(s, \mathbf{p}) = \sum_{i=1}^l \beta_i(s, \mathbf{p}) \mathbf{A}_i,$$

Minimal order (simplified)

[BENNER/GOYAL/P. '19]

$\text{rank} \left([\mathbf{O}^T \mathbf{A}_1 \mathbf{R} \quad \dots \quad \mathbf{O}^T \mathbf{A}_l \mathbf{R}] \right) = \begin{cases} \text{order of the minimal realization obtained by} \\ \text{removing unreachable and unobservable states} \end{cases}$

Dominant Reachable and Observable Projection (DROP)

- The proposed procedure remove uncontrollable and unobservable subspaces simultaneously.
- Neglecting small singular values leads to reduced-order models.

A time-delay demo system

$$\mathbf{H}(s) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2 e^{-s})^{-1} \mathbf{B}$$

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T,$$

A time-delay demo system

$$\mathbf{H}(s) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2 e^{-s})^{-1} \mathbf{B}$$

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T,$$

Let us construct, for $\sigma_i = [1, 2, 3, 4, 5, 6]$,

$$\mathbf{R} = [K(\sigma_1)^{-1}\mathbf{B} \quad \dots \quad K(\sigma_6)^{-1}\mathbf{B}],$$

$$\mathbf{O} = [K(\sigma_1)^{-T}\mathbf{C}^T \quad \dots \quad K(\sigma_6)^{-T}\mathbf{C}^T].$$

Numerical Examples

-Delay demo example-

A time-delay demo system

$$\mathbf{H}(s) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2 e^{-s})^{-1} \mathbf{B}$$

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T,$$

Let us construct, for $\sigma_i = [1, 2, 3, 4, 5, 6]$,

$$\mathbf{R} = [K(\sigma_1)^{-1}\mathbf{B} \quad \dots \quad K(\sigma_6)^{-1}\mathbf{B}],$$

$$\mathbf{O} = [K(\sigma_1)^{-T}\mathbf{C}^T \quad \dots \quad K(\sigma_6)^{-T}\mathbf{C}^T].$$

- $\text{rank}(\mathbf{R}) = 2, \text{rank}(\mathbf{O}) = 1.$ $\begin{pmatrix} \text{nonreachable} \\ \text{nonobservable} \end{pmatrix}$

A time-delay demo system

$$\mathbf{H}(s) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2 e^{-s})^{-1} \mathbf{B}$$

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T,$$

Let us construct, for $\sigma_i = [1, 2, 3, 4, 5, 6]$,

$$\mathbf{R} = [K(\sigma_1)^{-1}\mathbf{B} \quad \dots \quad K(\sigma_6)^{-1}\mathbf{B}],$$

$$\mathbf{O} = [K(\sigma_1)^{-T}\mathbf{C}^T \quad \dots \quad K(\sigma_6)^{-T}\mathbf{C}^T].$$

Then, using DROP, we get the projection matrices

$$\mathbf{V} = \mathbf{R}\mathbf{X}(:, 1) \text{ and } \mathbf{W} = \mathbf{O}\mathbf{Y}(:, 1).$$

- $\text{rank}(\mathbf{R}) = 2$, $\text{rank}(\mathbf{O}) = 1$. $\begin{pmatrix} \text{nonreachable} \\ \text{nonobservable} \end{pmatrix}$
- $\text{rank}([\mathbf{O}^T \mathbf{R} \quad \mathbf{O}^T \mathbf{A}_1 \mathbf{R} \quad \mathbf{O}^T \mathbf{A}_2 \mathbf{R}]) = 1$. (minimal realization order)

A time-delay demo system

$$\mathbf{H}(s) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2 e^{-s})^{-1} \mathbf{B}$$

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T,$$

Let us construct, for $\sigma_i = [1, 2, 3, 4, 5, 6]$,

$$\mathbf{R} = [K(\sigma_1)^{-1}\mathbf{B} \quad \dots \quad K(\sigma_6)^{-1}\mathbf{B}],$$

$$\mathbf{O} = [K(\sigma_1)^{-T}\mathbf{C}^T \quad \dots \quad K(\sigma_6)^{-T}\mathbf{C}^T].$$

- $\text{rank}(\mathbf{R}) = 2$, $\text{rank}(\mathbf{O}) = 1$. $\begin{pmatrix} \text{nonreachable} \\ \text{nonobservable} \end{pmatrix}$
- $\text{rank}([\mathbf{O}^T \mathbf{R} \quad \mathbf{O}^T \mathbf{A}_1 \mathbf{R} \quad \mathbf{O}^T \mathbf{A}_2 \mathbf{R}]) = 1$. $\begin{pmatrix} \text{(minimal} \\ \text{realization order)} \end{pmatrix}$

Then, using DROP, we get the projection matrices
 $\mathbf{V} = \mathbf{R}\mathbf{X}(:, 1)$ and $\mathbf{W} = \mathbf{O}\mathbf{Y}(:, 1)$.

The $\hat{\mathbf{H}}$ obtained using \mathbf{V} and \mathbf{W} satisfies

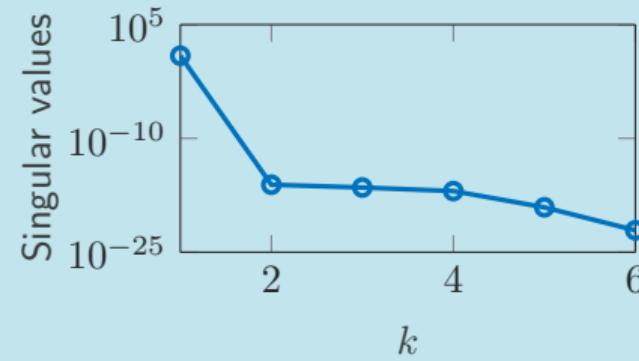
$$\mathbf{H}(s) = \hat{\mathbf{H}}(s), \forall s.$$

A time-delay demo system

$$\mathbf{H}(s) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2 e^{-s})^{-1} \mathbf{B}$$

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T,$$

Decay of singular values

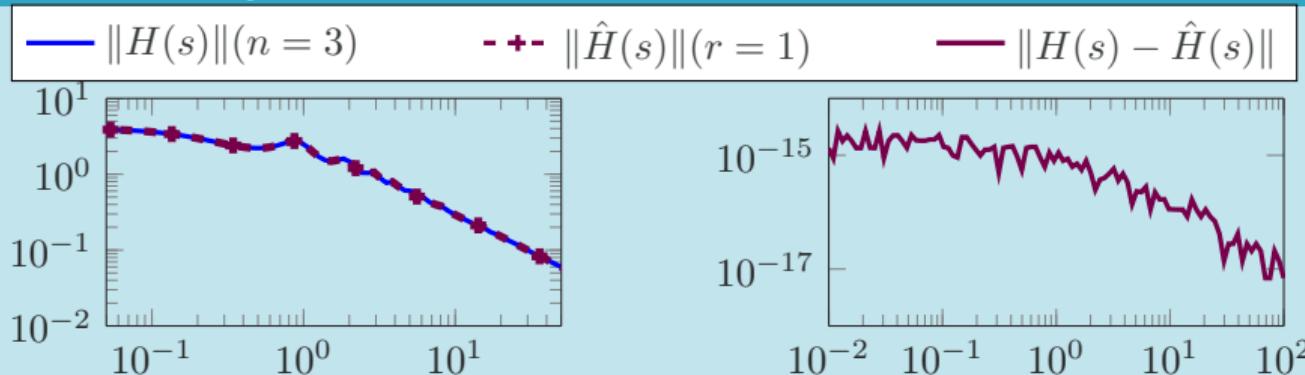


A time-delay demo system

$$\mathbf{H}(s) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2 e^{-s})^{-1} \mathbf{B}$$

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T,$$

Construction of a minimal system



A parametric demo dynamical system

$$\mathbf{H}(s, p) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - p\mathbf{A}_2)^{-1} \mathbf{B},$$

$$\mathbf{A}_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C}^T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

Numerical Examples

-Parametric demo example-

A parametric demo dynamical system

$$\mathbf{H}(s, p) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - p\mathbf{A}_2)^{-1} \mathbf{B},$$

$$\mathbf{A}_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C}^T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

- For $l = 20$ points (σ_i, \mathbf{p}_i) , let

$$\mathbf{R} = [K(\sigma_1, \mathbf{p}_1)^{-1} \mathbf{B} \quad \dots \quad K(\sigma_l, \mathbf{p}_l)^{-1} \mathbf{B}],$$

$$\mathbf{O} = [K(\sigma_1, \mathbf{p}_1)^{-T} \mathbf{C}^T \quad \dots \quad K(\sigma_l, \mathbf{p}_l)^{-T} \mathbf{C}^T].$$

A parametric demo dynamical system

$$\mathbf{H}(s, p) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - p\mathbf{A}_2)^{-1} \mathbf{B},$$

$$\mathbf{A}_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C}^T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

- For $l = 20$ points (σ_i, \mathbf{p}_i) , let

$$\mathbf{R} = [K(\sigma_1, \mathbf{p}_1)^{-1} \mathbf{B} \quad \dots \quad K(\sigma_l, \mathbf{p}_l)^{-1} \mathbf{B}],$$

$$\mathbf{O} = [K(\sigma_1, \mathbf{p}_1)^{-T} \mathbf{C}^T \quad \dots \quad K(\sigma_l, \mathbf{p}_l)^{-T} \mathbf{C}^T].$$

- $\text{rank}([\mathbf{O}^T \mathbf{R} \quad \mathbf{O}^T \mathbf{A}_1 \mathbf{R} \quad \mathbf{O}^T \mathbf{A}_2 \mathbf{R}]) = 2.$

Numerical Examples

-Parametric demo example-

A parametric demo dynamical system

$$\mathbf{H}(s, p) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - p\mathbf{A}_2)^{-1} \mathbf{B},$$

$$\mathbf{A}_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C}^T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

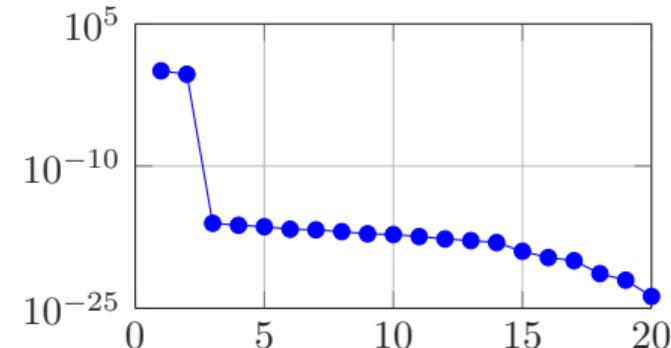
- For $l = 20$ points (σ_i, \mathbf{p}_i) , let

$$\mathbf{R} = [K(\sigma_1, \mathbf{p}_1)^{-1} \mathbf{B} \quad \dots \quad K(\sigma_l, \mathbf{p}_l)^{-1} \mathbf{B}],$$

$$\mathbf{O} = [K(\sigma_1, \mathbf{p}_1)^{-T} \mathbf{C}^T \quad \dots \quad K(\sigma_l, \mathbf{p}_l)^{-T} \mathbf{C}^T].$$

- $\text{rank}([\mathbf{O}^T \mathbf{R} \quad \mathbf{O}^T \mathbf{A}_1 \mathbf{R} \quad \mathbf{O}^T \mathbf{A}_2 \mathbf{R}]) = 2.$
- Compute projectors \mathbf{V} and \mathbf{W} and $\hat{\mathbf{H}}(s, p).$
- Then, $\mathbf{H}(s, p) = \hat{\mathbf{H}}(s, p).$

Decay of Singular values



Numerical Examples

-Parametric demo example-

A parametric demo dynamical system

$$\mathbf{H}(s, p) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - p\mathbf{A}_2)^{-1} \mathbf{B},$$

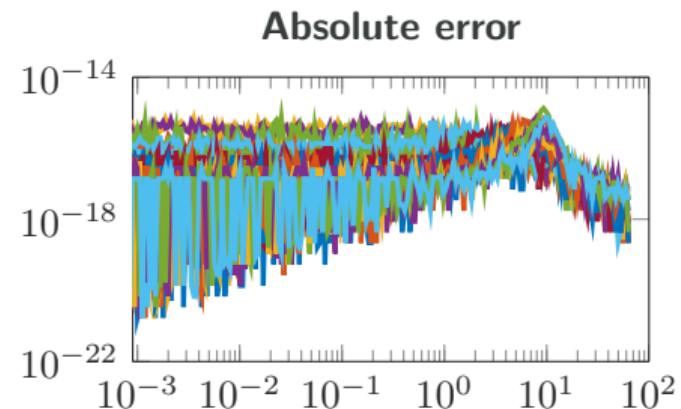
$$\mathbf{A}_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C}^T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

- For $l = 20$ points (σ_i, \mathbf{p}_i) , let

$$\mathbf{R} = [K(\sigma_1, \mathbf{p}_1)^{-1} \mathbf{B} \quad \dots \quad K(\sigma_l, \mathbf{p}_l)^{-1} \mathbf{B}],$$

$$\mathbf{O} = [K(\sigma_1, \mathbf{p}_1)^{-T} \mathbf{C}^T \quad \dots \quad K(\sigma_l, \mathbf{p}_l)^{-T} \mathbf{C}^T].$$

- $\text{rank}([\mathbf{O}^T \mathbf{R} \quad \mathbf{O}^T \mathbf{A}_1 \mathbf{R} \quad \mathbf{O}^T \mathbf{A}_2 \mathbf{R}]) = 2$.
- Compute projectors \mathbf{V} and \mathbf{W} and $\hat{\mathbf{H}}(s, p)$.
- Then, $\mathbf{H}(s, p) = \hat{\mathbf{H}}(s, p)$.



Delay example

[BEATTIE/GUGERCIN '09]

$$\begin{aligned}\mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{A}_\tau \mathbf{x}(t - \tau) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t).\end{aligned}$$

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A}_1 - \mathbf{A}_\tau e^{-s\tau})\mathbf{B}$$

- Full order model $n = 500$ and $\tau = 1$.
- To employ the proposed methods, we consider 100 points on the imaginary axis.

Delay example

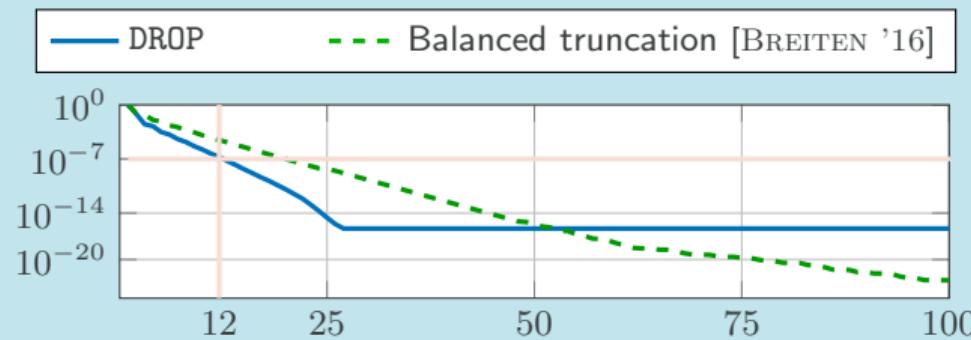
[BEATTIE/GUGERCIN '09]

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{A}_\tau \mathbf{x}(t - \tau) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t).\end{aligned}$$

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A}_1 - \mathbf{A}_\tau e^{-s\tau})\mathbf{B}$$

- Full order model $n = 500$ and $\tau = 1$.
- To employ the proposed methods, we consider 100 points on the imaginary axis.

Decay of singular values



Delay example

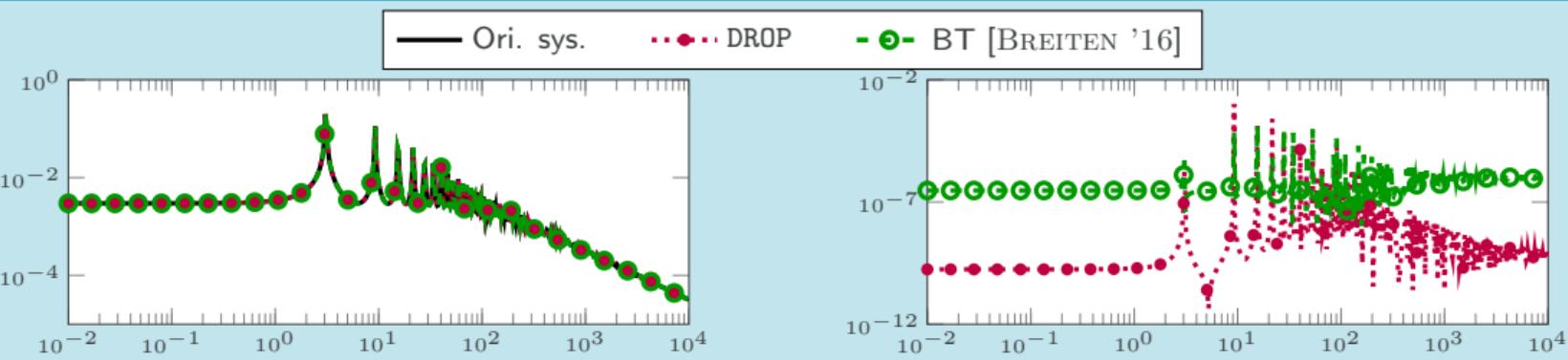
[BEATTIE/GUGERCIN '09]

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{A}_\tau \mathbf{x}(t - \tau) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t).\end{aligned}$$

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A}_1 - \mathbf{A}_\tau e^{-s\tau})\mathbf{B}$$

- Full order model $n = 500$ and $\tau = 1$.
- To employ the proposed methods, we consider 100 points on the imaginary axis.

Reduced system of order $r = 12$



Fractional Maxwell equations.

[FENG/BENNER '08]

$$\mathbf{H}(s) = s\mathbf{B}^T \left(s^2\mathbf{I} - \frac{1}{\sqrt{s}}\mathbf{D} + \mathbf{A} \right)^{-1} \mathbf{B},$$

- Full order model $n = 29,295$. Frequency range is $\mathcal{F} := [4e9, 8e9]\text{Hz}$.
- To employ the proposed methods, we consider 50 points on the imaginary axis.

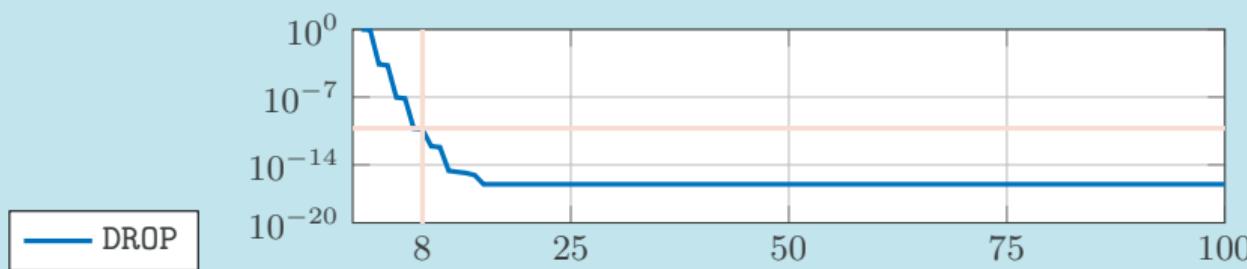
Fractional Maxwell equations.

[FENG/BENNER '08]

$$\mathbf{H}(s) = s\mathbf{B}^T \left(s^2\mathbf{I} - \frac{1}{\sqrt{s}}\mathbf{D} + \mathbf{A} \right)^{-1} \mathbf{B},$$

- Full order model $n = 29,295$. Frequency range is $\mathcal{F} := [4e9, 8e9]$ Hz.
- To employ the proposed methods, we consider 50 points on the imaginary axis.

Decay of singular values



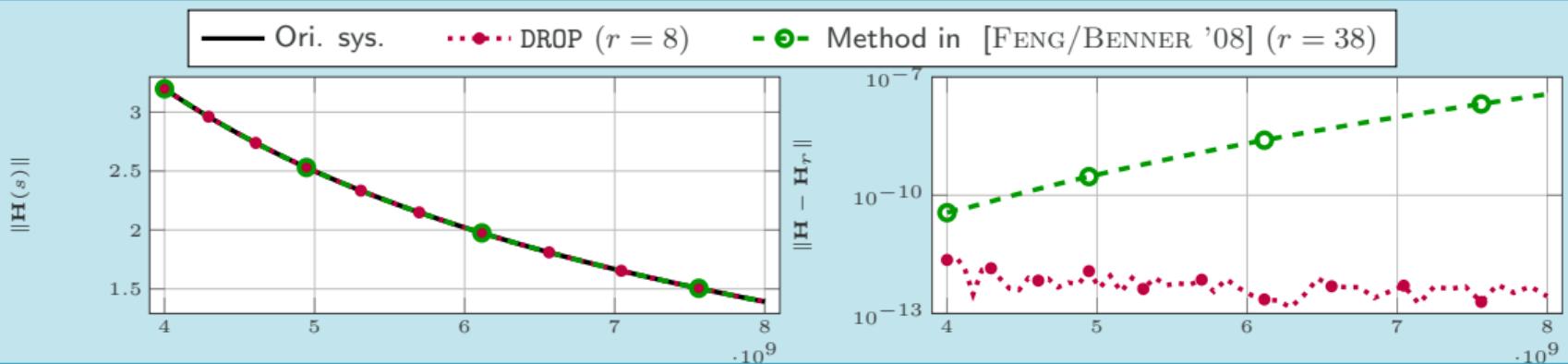
Fractional Maxwell equations.

[FENG/BENNER '08]

$$\mathbf{H}(s) = s\mathbf{B}^T \left(s^2\mathbf{I} - \frac{1}{\sqrt{s}}\mathbf{D} + \mathbf{A} \right)^{-1} \mathbf{B},$$

- Full order model $n = 29,295$. Frequency range is $\mathcal{F} := [4e9, 8e9]$ Hz.
- To employ the proposed methods, we consider 50 points on the imaginary axis.

Reduced system of order $r = 12$



Parametric Butterfly Gyroscope

[MORWIKI, MODIFIED GYROSCOPE]

$$\mathbf{M}(d)\ddot{\mathbf{x}}(t) + \mathbf{D}(d, \theta)\dot{\mathbf{x}}(t) + \mathbf{K}(\theta) = \mathbf{B}\mathbf{u}(t),$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

where

$$\mathbf{M}(d) = \mathbf{M}_1 + d\mathbf{M}_2,$$

$$\mathbf{D}(d, \theta) = \theta (\mathbf{D}_1 + d\mathbf{D}_2),$$

$$\mathbf{K}(d) = \mathbf{T}_1 + \frac{1}{d}\mathbf{T}_2 + d\mathbf{T}_3.$$

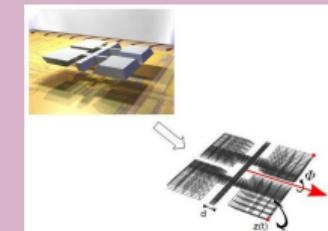


Figure: Semantic gyroscope diagram.

Parameters and frequency range:

$$\theta \in [10^{-5}, 10^{-7}], d \in [1, 2] \quad freq \in [0.025, 40].$$

Order of the system: $n = 17,913$.

Numerical Examples

-Parametric Butterfly Gyroscope-

Parametric Butterfly Gyroscope

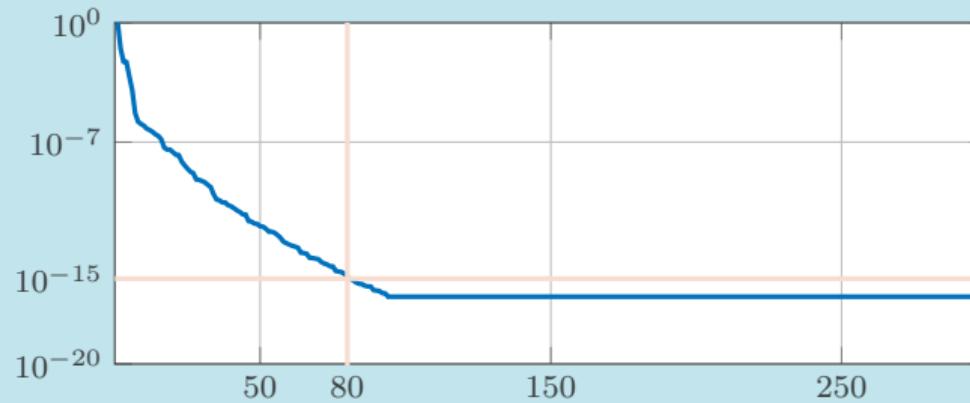


Figure: Gyro example: relative decay of the singular values obtained using the proposed method.

- We take 500 points for frequency s in the logarithmic way and the same number of random points for parameter $\mathbf{p} = [d, \theta]^T$ in the considered range.

Parametric Butterfly Gyroscope

— Orig. sys. ($n = 17,913$) - - - DROP ($r = 80$) ····· Method in [FENG, ANTOULAS, BENNER 17'] ($r = 210$)

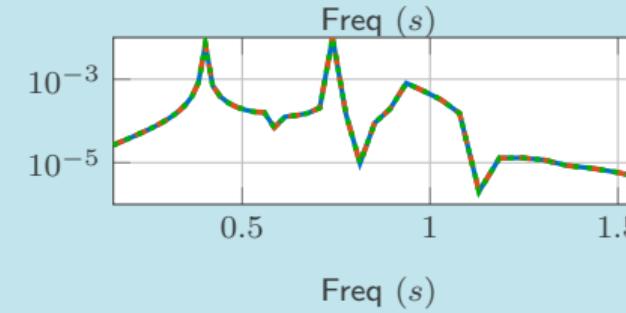
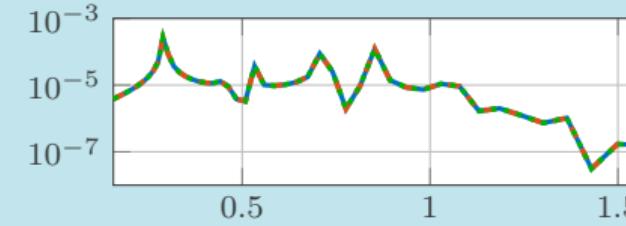
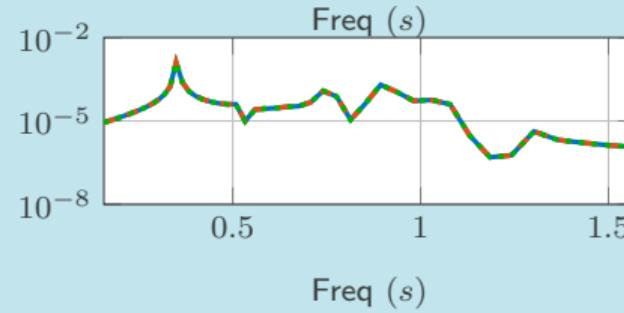
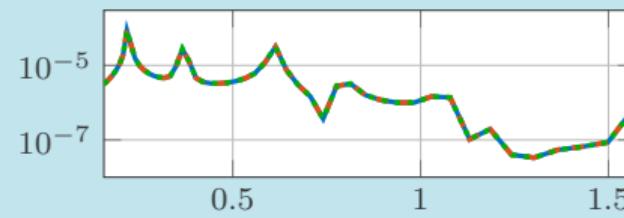


Figure: $\mathbf{p}^{(1)} : (1.00, 10^{-7})$, $\mathbf{p}^{(2)} : (1.33, 4.64 \cdot 10^{-7})$, $\mathbf{p}^{(3)} : (1.67, 2.15 \cdot 10^{-6})$, $\mathbf{p}^{(4)} : (2.00, 10^{-5})$.

Parametric Butterfly Gyroscope

— Orig. sys. ($n = 17,913$) - - - DROP ($r = 80$) ····· Method in [FENG, ANTOULAS, BENNER 17'] ($r = 210$)

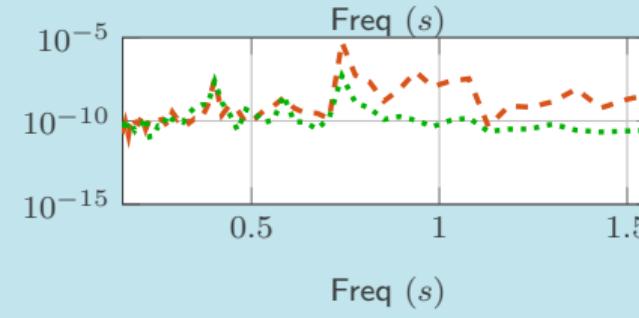
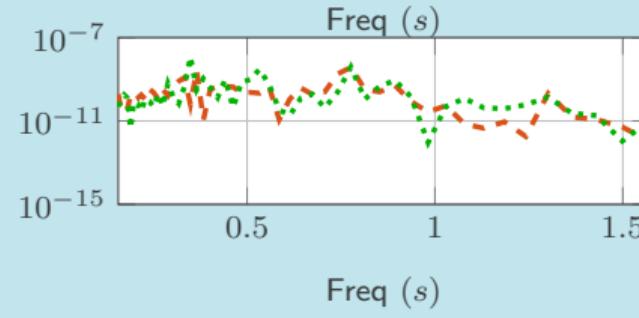
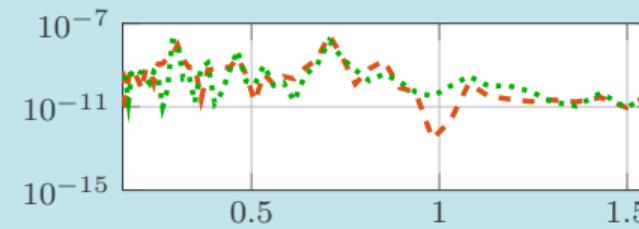
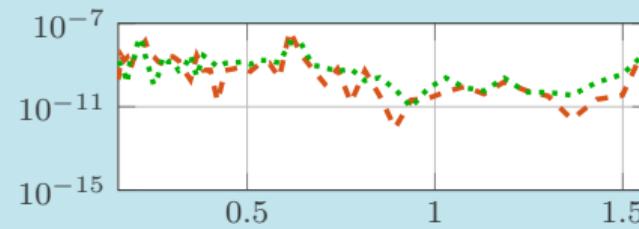


Figure: $\mathbf{p}^{(1)} : (1.00, 10^{-7})$, $\mathbf{p}^{(2)} : (1.33, 4.64 \cdot 10^{-7})$, $\mathbf{p}^{(3)} : (1.67, 2.15 \cdot 10^{-6})$, $\mathbf{p}^{(4)} : (2.00, 10^{-5})$.



Outlook

Contributions of the talk

- Minimal realization and **reduced-order modeling** for structured linear systems.
- **DROP** (Dominant Reachable and Observable Projection) algorithm.
- **Computational aspects** in a large-scale setting (low-rank factors, randomized SVDs).
- Extend results to parametric systems.
- Application to benchmarks examples.

Contributions of the talk

- Minimal realization and **reduced-order modeling** for structured linear systems.
- **DROP** (Dominant Reachable and Observable Projection) algorithm.
- **Computational aspects** in a large-scale setting (low-rank factors, randomized SVDs).
- Extend results to parametric systems.
- Application to benchmarks examples.

Open questions and future work

- Errors estimators and parameter choice.
- Stability of reduced-order systems.

Reference: Benner, P., Goyal, P., & Pontes Duff, I. (2019). Identification of Dominant Subspaces for Linear Structured Parametric Systems and Model Reduction. arXiv preprint arXiv:1910.13945.